Serie 6 - Fixed Points

Exercise 1

We represent non-negative integers with the following Lambda expressions:

\[
\begin{align*}
0 & \equiv \lambda f . \lambda x . x \\
1 & \equiv \lambda f . \lambda x . fx \\
2 & \equiv \lambda f . \lambda x . f(fx) \\
& \vdots \\
n & \equiv \lambda f . \lambda x . f^n x
\end{align*}
\]

Suppose you have defined the function \texttt{if} and the operations \texttt{add}, \texttt{pred} and \texttt{isZero}. Consider the following recursive (and hence not valid) definition for the multiplication:

\[
times = \lambda n_1 . \lambda n_2 . \texttt{if} (\texttt{isZero} \ n_1) \ 0 \ (\texttt{add} \ n_2 \ (\times \ (\texttt{pred} \ n_1) \ n_2))
\]

If we abstract the name \texttt{times}, we get the new expression:

\[
t = \lambda f . \lambda n_1 . \lambda n_2 . \texttt{if} (\texttt{isZero} \ n_1) \ 0 \ (\texttt{add} \ n_2 \ (f \ (\texttt{pred} \ n_1) \ n_2))
\]

By the FP theorem we know that \((Y \ t)\) is a non-recursive equivalent of the above \texttt{times} definition.

The exercise: write down the reduction sequence to demonstrate that

\[
(((Y \ t) \ 1) \ k) \rightarrow k.
\]

Exercise 2

We can represent lists and list operators with the following Lambda expressions:

\[
\begin{align*}
nil & = \lambda f . \texttt{true} \\
null & = \lambda l . l \ (\lambda h . \lambda t . \texttt{false}) \\
\texttt{cons} & = \lambda h . \lambda t . \lambda f . fht \\
\texttt{head} & = \lambda l . l \ (\lambda h . \lambda t . h) \\
\texttt{tail} & = \lambda l . l \ (\lambda h . \lambda t . t)
\end{align*}
\]

Example: the list \([1, 2, 3]\) is represented by the \(\lambda\)-expression \texttt{cons 1 (cons 2 (cons 3 nil))}.
To do:

1. Translate the following definition into a non-recursive form:

\[
\text{append} = \lambda l_1 . \lambda l_2 . \text{if} (\text{null } l_1) l_2 (\text{cons} (\text{head } l_1) (\text{append} (\text{tail } l_1) l_2))
\]

2. Test your result by appending list \(L_2\) to list \(L_1\), which are defined below:

\[L_1 = \text{cons } 1 \text{ (cons } 2 \text{ nil) and } L_2 = \text{cons } 3 \text{ nil}\]