# S7057 Programmiersprachen

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Sommersemester 2001

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# 1. Programming Languages

| | |
|---|---|
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# Sources

**Text:**

- Kenneth C. Louden, *Programming Languages: Principles and Practice*, PWS Publishing (Boston), 1993.

**Other Sources:**

- *PostScript® Language Tutorial and Cookbook*, Adobe Systems Incorporated, Addison-Wesley, 1985
- Paul Hudak, “Conception, Evolution, and Application of Functional Programming Languages,” *ACM Computing Surveys* 21/3, pp 359-411.
- Clocksin and Mellish, *Programming in Prolog*, Springer Verlag, 1981.

# Schedule

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| 1. | 03 - 27 | Introduction |
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# Themes Addressed in this Course

## Paradigms

- What computational paradigms are supported by modern, high-level programming languages?
- How well do these paradigms match classes of programming problems?

## Abstraction

- How do different languages abstract away from the low-level details of the underlying hardware implementation?
- How do different languages support the specification of software abstractions needed for a specific task?

...

# Themes Addressed in this Course ...

**Types**

- How do type systems help in the construction of flexible, reliable software?

**Semantics**

- How can one formalize the meaning of a programming language?
- How can semantics aid in the implementation of a programming language?

# What is a Programming Language?

- ☞ A formal language for describing computation
- ☞ A "user interface" to a computer
- ☞ "Turing tar pit" — equivalent computational power
- ☞ Programming paradigms — different expressive power
- ☞ Syntax + semantics
- ☞ Compiler, or interpreter, or translator

*"A programming language is a notational system for describing computation in a machine-readable and human-readable form."*

*— Louden*

# Generations of Programming Languages

**1GL:** machine codes

**2GL:** symbolic assemblers

**3GL:** (machine independent) imperative languages (FORTRAN, Pascal ...)

**4GL:** domain specific application generators

*Each generation is at a higher level of abstraction*

# How do Programming Languages Differ?

**Common Constructs:**

- ☞ basic data types (numbers, etc.); variables; expressions; statements; keywords; control constructs; procedures; comments; errors ...

**Uncommon Constructs:**

- ☞ type declarations; special types (strings, arrays, matrices, ...); sequential execution; concurrency constructs; packages/modules; objects; general functions; generics; modifiable state; ...

# Programming Paradigms

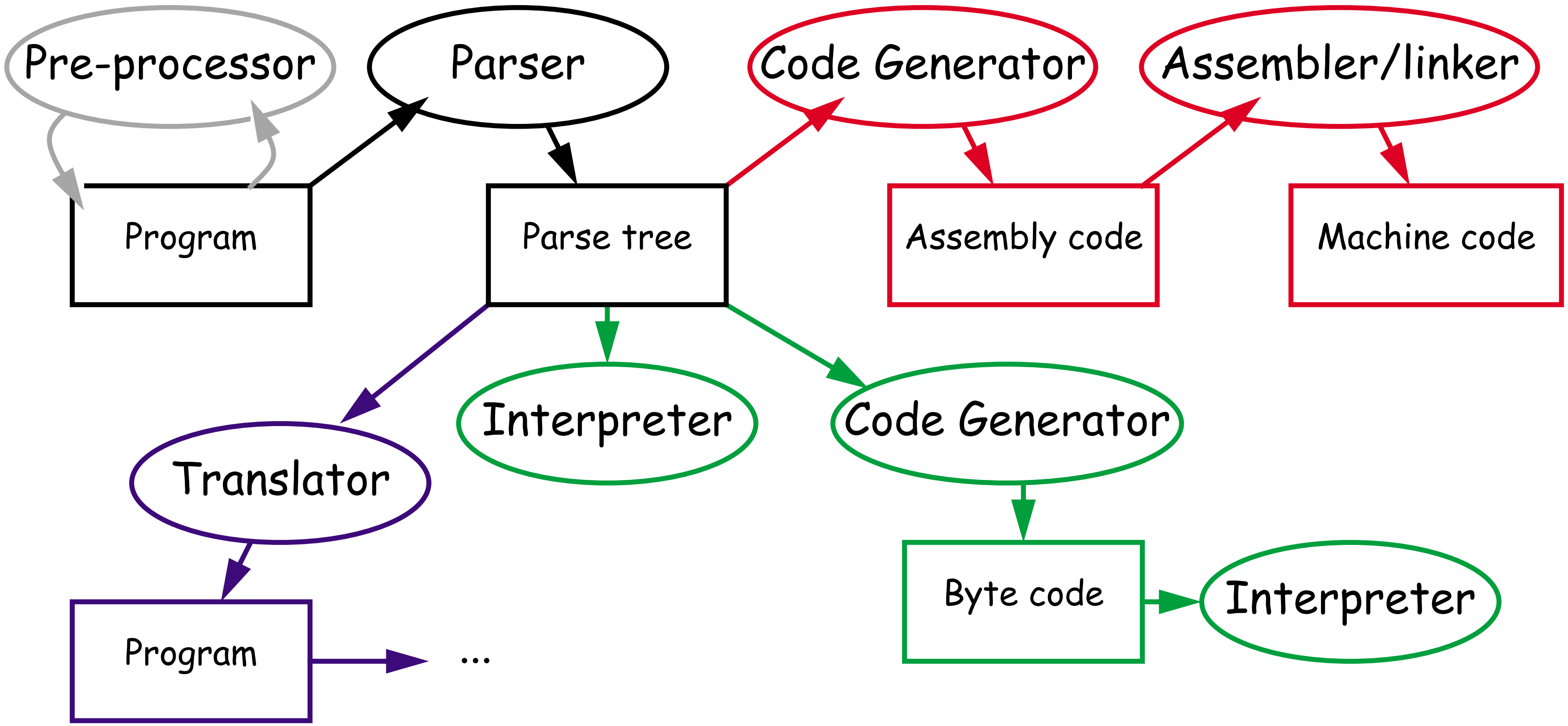
*A programming language is a problem-solving tool.*

| | |
|---|---|
| *Imperative style:* | program = algorithms + data<br>*good for decomposition* |
| *Functional style:* | program = functions $\circ$ functions<br>*good for reasoning* |
| *Logic programming style:* | program = facts + rules<br>*good for searching* |
| *Object-oriented style:* | program = objects + messages<br>*good for encapsulation* |

*Other styles and paradigms:* blackboard, pipes and filters, constraints, lists, ...

# Compilers and Interpreters

Compilers and interpreters have similar front-ends, but have different back-ends:




Details will differ, but the general scheme remains the same ...

# A Brief Chronology

**Early 1950s** "order codes" (primitive assemblers)

| | | |
|---|---|---|
| 1957 | FORTRAN | the first *high-level* programming language (3GL is invented) |
| 1958 | ALGOL | the first *modern, imperative* language |
| 1960 | LISP, COBOL | |
| 1962 | APL, SIMULA | the birth of *OOP* (SIMULA) |
| 1964 | BASIC, PL/I | |
| 1966 | ISWIM | first modern *functional* language (a proposal) |
| 1970 | Prolog | *logic* programming is born |
| 1972 | C | *the* systems programming language |
| 1975 | Pascal, Scheme | two teaching languages |

| | | |
|---|---|---|
| 1978 | CSP | |
| 1978 | FP | |
| 1980 | dBASE II | |
| 1983 | Smalltalk-80, Ada | OOP is reinvented |
| 1984 | Standard ML | FP becomes mainstream (?) |
| 1986 | C++, Eiffel | OOP is reinvented (again) |
| 1988 | CLOS, Oberon, Mathematica | |
| 1990 | Haskell | FP is reinvented |
| 1995 | Java | OOP is reinvented for the internet |

# Fortran

**History**

John Backus (1953) sought to write programs in *conventional mathematical notation*, and generate code comparable to good assembly programs.

- No language design effort
  (made it up as they went along)
- Most effort spent on code generation and optimization
- FORTRAN I released April 1957; working by April 1958
- Current standards are FORTRAN 77 and FORTRAN 90

...

# Fortran ...

**Innovations**

- *Symbolic notation* for subroutines and functions
- Assignments to variables of complex expressions
- DO loops
- Comments
- Input/output formats
- Machine-independence

**Successes**

- Easy to learn; high level
- Promoted by IBM; addressed large user base (scientific computing)

# ALGOL 60

## History

- Committee of PL experts formed in 1955 to design universal, machine-independent, algorithmic language
- First version (ALGOL 58) never implemented; criticisms led to ALGOL 60

...

# ALGOL 60 ...

**Innovations**

- *BNF* (Backus-Naur Form) introduced to define syntax (led to syntax-directed compilers)
- First *block-structured* language; variables with local scope
- *Structured* control statements
- *Recursive* procedures
- Variable size arrays

**Successes**

- Highly influenced design of other PLs but never displaced FORTRAN

# COBOL

## History

- Designed by committee of US computer manufacturers
- Targeted business applications
- Intended to be readable by managers (!)

## Innovations

- Separate descriptions of environment, data, and processes

## Successes

- Adopted as *de facto* standard by US DOD
- Stable standard for 25 years
- Still the *most widely used PL* for business applications (!)

# 4GLs

**"Problem-oriented" languages**

- PLs for "non-programmers"
- *Very High Level* (VHL) languages for *specific* problem domains

**Classes of 4GLs (no clear boundaries)**

- Report Program Generator (RPG)
- Application generators
- Query languages
- Decision-support languages

**Successes**

- Highly popular, but generally *ad hoc*

# PL/I

## History

- Designed by committee of IBM and users (early 1960s)
- Intended as (large) *general-purpose language* for broad classes of applications

## Innovations

- Support for *concurrency* (but not synchronization)
- *Exception-handling* by on conditions

## Successes

- Achieved both run-time efficiency and flexibility (at expense of complexity)
- First "complete" general purpose language

# Interactive Languages

Made possible by advent of *time-sharing* systems (early 1960s through mid 1970s).

**BASIC**

- Developed at Dartmouth College in mid 1960s
- Minimal; easy to learn
- Incorporated basic O/S commands (NEW, LIST, DELETE, RUN, SAVE)

...

# Interactive Languages ...

**APL**

- Developed by Ken Iverson for *concise* description of numerical algorithms
- Large, non-standard alphabet (52 characters in addition to alphanumerics)
- Primitive objects are *arrays* (lists, tables or matrices)
- *Operator-driven* (power comes from composing array operators)
- No operator precedence (statements parsed right to left)

# Special-Purpose Languages

**SNOBOL**

- First successful *string manipulation* language
- Influenced design of text editors more than other PLs
- String operations: *pattern-matching* and *substitution*
- Arrays and associative arrays (tables)
- Variable-length strings

...

# Special-Purpose Languages ...

## Lisp

- Performs computations on symbolic expressions
- *Symbolic expressions* are represented as *lists*
- Small set of constructor/selector operations to create and manipulate lists
- *Recursive* rather than iterative control
- No distinction between *data* and *programs*
- First PL to implement storage management by *garbage collection*
- Affinity with *lambda calculus*

# Functional Languages

**ISWIM (If you See What I Mean)**

- Peter Landin (1966) — paper proposal

**FP**

- John Backus (1978) — Turing award lecture

**ML**

- Edinburgh
- initially designed as *meta-language* for theorem proving
- Hindley-Milner *type inference*
- "non-pure" functional language (with assignments/side effects)

**Miranda, Haskell**

- "*pure*" functional languages with "*lazy evaluation*"

# Prolog

## History

- Originated at U. Marseilles (early 1970s), and compilers developed at Marseilles and Edinburgh (mid to late 1970s)

## Innovations

- *Theorem proving* paradigm
- Programs as sets of clauses: *facts*, *rules* and *questions*
- Computation by "*unification*"

## Successes

- Prototypical logic programming language
- Used in Japanese Fifth Generation Initiative

# Object-Oriented Languages

## History

- **Simula** was developed by Nygaard and Dahl (early 1960s) in Oslo as a language for simulation programming, by adding *classes* and *inheritance* to ALGOL 60
- **Smalltalk** was developed by Xerox PARC (early 1970s) to drive graphic workstations

...

# Object-Oriented Languages ...

**Innovations**

- *Encapsulation* of data and operations (contrast ADTs)
- *Inheritance* to share behaviour and interfaces

**Successes**

- Smalltalk project pioneered OO *user interfaces*
- Large commercial impact since mid 1980s
- Countless new languages: C++, Objective C, Eiffel, Beta, Oberon, Self, Perl 5, Python, Java, Ada 95 ...

# Scripting Languages

## History

- Countless "shell languages" and "command languages" for operating systems and configurable applications
- **Unix shell** (ca. 1971) developed as user shell and scripting tool
- **HyperTalk** (1987) was developed at Apple to script HyperCard stacks
- **TCL** (1990) developed as embedding language and scripting language for X windows applications (via Tk)
- **Perl** (~1990) became de facto web scripting language

...

# Scripting Languages ...

**Innovations**

- Pipes and filters (Unix shell)
- Generalized embedding/command languages (TCL)

**Successes**

- Unix Shell, awk, emacs, HyperTalk, AppleTalk, TCL, Python, Perl, VisualBasic ...

# What you should know!

- ✎ What, exactly, is a programming language?
- ✎ How do compilers and interpreters differ?
- ✎ Why was FORTRAN developed?
- ✎ What were the main achievements of ALGOL 60?
- ✎ Why do we call Pascal a "Third Generation Language"?
- ✎ What is a "Fourth Generation Language"?

# Can you answer these questions?

- *Why are there so many programming languages?*
- *Why are FORTRAN and COBOL still important programming languages?*
- *Which language should you use to implement a spelling checker?*
  *A filter to translate upper-to-lower case?*
  *A theorem prover?*
  *An address database?*
  *An expert system?*
  *A game server for initiating chess games on the internet?*
  *A user interface for a network chess client?*

# 2. Stack-based Programming

**Overview**

- PostScript objects, types and stacks
- Arithmetic operators
- Graphics operators
- Procedures and variables
- Arrays and dictionaries

**References:**


- *PostScript® Language Tutorial and Cookbook*, Adobe Systems Incorporated, Addison-Wesley, 1985
- *PostScript® Language Reference Manual*, Adobe Systems Incorporated, second edition, Addison-Wesley, 1990

# PostScript

PostScript "is a simple interpretive programming language ... to describe the appearance of text, graphical shapes, and sampled images on printed or displayed pages."

- introduced in 1985 by Adobe
- display standard now supported by all major printer vendors
- simple, stack-based programming language
- minimal syntax
- large set of built-in operators
- PostScript programs are usually generated from applications, rather than hand-coded

# Postscript variants

**Level 1:**

- the original 1985 PostScript

**Level 2:**

- additional support for dictionaries, memory management ...

**Display PostScript:**

- special support for screen display

**Level 3:**

- the current incarnation with “workflow” support

# Syntax

| | |
|---|---|
| *Comments:* | from "%" to next newline or formfeed<br>`% This is a comment` |
| *Numbers:* | signed integers, reals and radix numbers<br>`123 -98 0 +17 -.002 34.5`<br>`123.6e10 1E-5 8#1777 16#FFE 2#1000` |
| *Strings:* | text in *parentheses* or hexadecimal in *angle brackets* (Special characters are escaped: \n \t \( \) \\ …) |
| *Names:* | tokens consisting of "regular characters" but which aren't numbers<br>`abc Offset $$ 23A 13-456 a.b`<br>`$MyDict @pattern` |

| | |
|---|---|
| *Literal names:* | start with *slash* |
| | `/buffer /proc` |
| *Arrays:* | enclosed in *square brackets* |
| | `[ 123 /abc (hello) ]` |
| *Procedures:* | enclosed in *curly brackets* |
| | `{ add 2 div }`<br>`% add top two stack items and divide by 2` |

# Semantics

A PostScript program is a *sequence of tokens*, representing *typed objects*, that is interpreted to manipulate the *display* and four *stacks* that represent the execution state of a PostScript program:

| | |
|---|---|
| *Operand stack:* | holds (arbitrary) *operands* and *results* of PostScript operators |
| *Dictionary stack:* | holds only *dictionaries* where keys and values may be stored |
| *Execution stack:* | holds *executable objects* (e.g. procedures) in stages of execution |
| *Graphics state stack:* | keeps track of current *coordinates* etc. |

# Object types

Every object is either *literal* or *executable:*

*Literal objects* are *pushed* on the operand stack:

- integers, reals, string constants, literal names, arrays, procedures

*Executable objects* are *interpreted:*

- built-in operators
- names bound to procedures (in the current dictionary context)

*Simple Object Types* are copied by *value*

- boolean, fontID, integer, name, null, operator, real ...

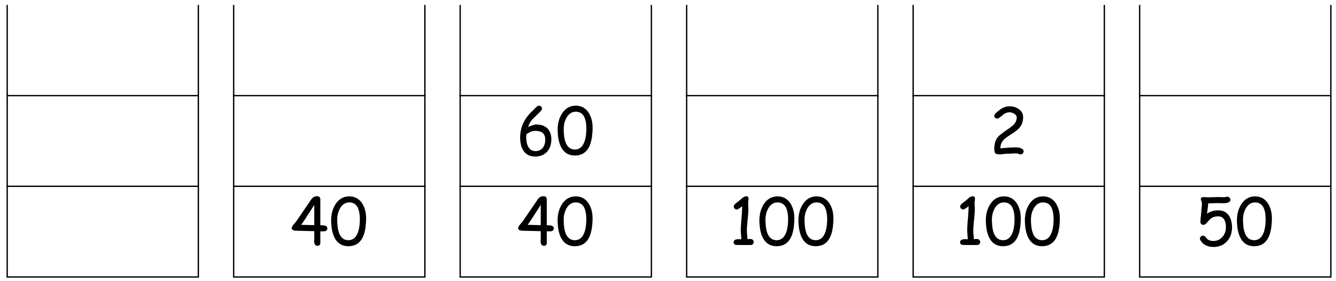
*Composite Object Types* are copied by *reference*

- array, dictionary, string ...

# The operand stack

Compute the average of 40 and 60:

```
40 60 add 2 div
```




At the end, the result is left on the top of the operand stack.

# Stack and arithmetic operators

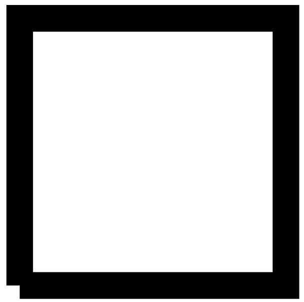
| Stack | Op | New Stack | Function |
|---|---|---|---|
| $num_1$ $num_2$ | `add` | sum | *$num_1 + num_2$* |
| $num_1$ $num_2$ | `sub` | difference | *$num_1 - num_2$* |
| $num_1$ $num_2$ | `mul` | product | *$num_1 * num_2$* |
| $num_1$ $num_2$ | `div` | quotient | *$num_1 / num_2$* |
| $int_1$ $int_2$ | `idiv` | quotient | integer divide |
| $int_1$ $int_2$ | `mod` | remainder | *$int_1$* mod *$int_2$* |
| num den | `atan` | angle | arctangent of *num/den* |
| any | `pop` | - | discard top element |
| $any_1$ $any_2$ | `exch` | $any_2$ $any_1$ | exchange top two elements |
| any | `dup` | any any | duplicate top element |
| $any_1$ ... $any_n$ n | `copy` | $any_1$ ... $any_n$ $any_1$ ... $any_n$ | duplicate top *n* elements |
| $any_n$ ... $any_0$ n | `index` | $any_n$ ... $any_0$ $any_n$ | duplicate *n+1*th element |

*and many others ...*

# Drawing a Box

"A *path* is a set of straight lines and curves that define a region to be filled or a trajectory that is to be drawn on the *current page*."

```
newpath                % clear the current drawing path
100 100 moveto         % move to (100,100)
100 200 lineto         % draw a line to (100,200)
200 200 lineto
200 100 lineto
100 100 lineto
10 setlinewidth        % set width for drawing
stroke                 % draw along current path
showpage               % and display current page
```



# Path construction operators

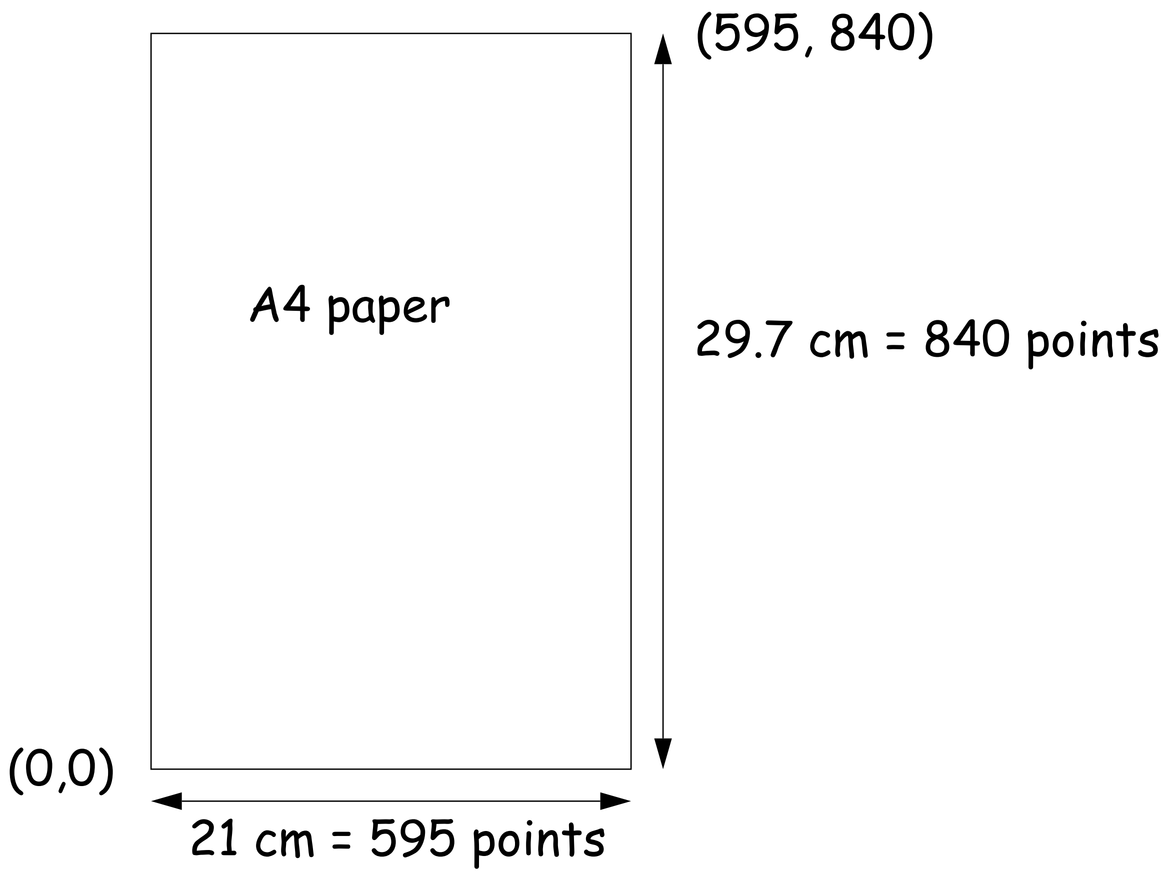
| | | | |
|---|---|---|---|
| - | `newpath` | - | initialize current path to be empty |
| - | `currentpoint` | x y | return current coordinates |
| x y | `moveto` | - | set current point to *(x, y)* |
| dx dy | `rmoveto` | - | relative moveto |
| x y | `lineto` | - | append straight line to *(x, y)* |
| dx dy | `rlineto` | - | relative lineto |
| x y r $ang_1$ $ang_2$ | `arc` | - | append counterclockwise arc |
| - | `closepath` | - | connect subpath back to start |
| - | `fill` | - | fill current path with current colour |
| - | `stroke` | - | draw line along current path |
| - | `showpage` | - | output and reset current page |

*Others:* arcn, arcto, curveto, rcurveto, flattenpath, ...

# Coordinates

Coordinates are measured in *points:*

*72 points = 1 inch = 2.54 cm.*



# Hello World

Before you can print text, you must (1) *look up* the desired font, (2) *scale it* to the required size, and (3) *set it* to be the *current font*.

```
/Times-Roman findfont % look up Times Roman font
  18 scalefont          % scale it to 18 points
  setfont               % set this to be the current font
100 500 moveto          % go to coordinate (100, 500)
(Hello world) show      % draw the string “Hello world”
showpage                % render the current page
```

# Character and font operators

| | | | |
|---|---|---|---|
| key | `findfont` | font | return font dict identified by *key* |
| font scale | `scalefont` | font′ | scale *font* by *scale* to produce *font′* |
| font | `setfont` | - | set font dictionary |
| - | `currentfont` | font | return current font |
| string | `show` | - | print *string* |
| string | `stringwidth` | $w_x$ $w_y$ | width of *string* in current font |

*Others:* definefont, makefont, FontDirectory, StandardEncoding ....

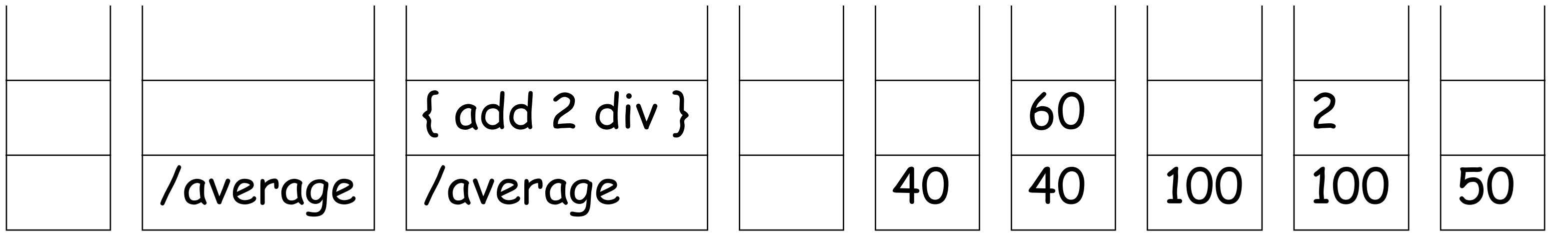
# Procedures and Variables

Variables and procedures are defined by binding *names* to *literal* or *executable* objects.

| | | | |
|---|---|---|---|
| key value | def | - | associate *key* and *value* in current dictionary |

*Define a general procedure to compute averages:*

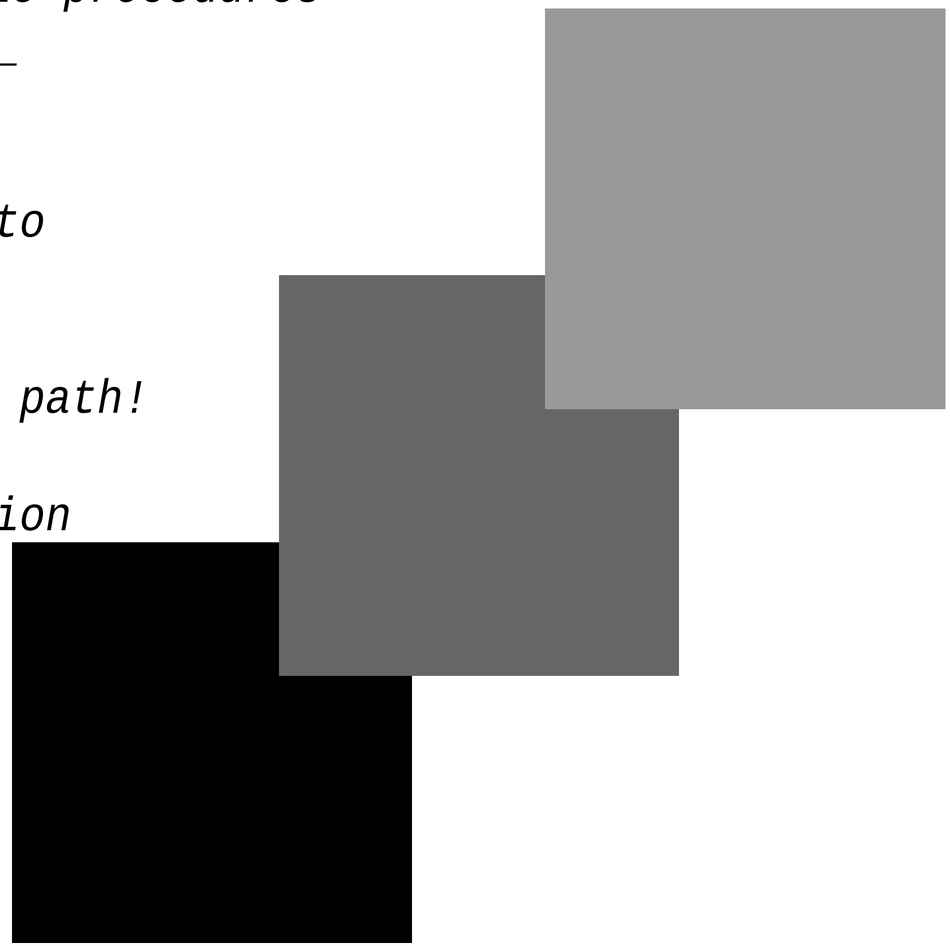
```
/average { add 2 div } def
% bind the name "average" to "{ add 2 div }"
40 60 average
```



# A Box procedure

Most PostScript programs consist of a *prologue* and a *script*.

```
% Prologue -- application specific procedures
/box {                % grey x y -> __
   newpath
   moveto             % x y -> __
   0 150 rlineto      % relative lineto
   150 0 rlineto
   0 -150 rlineto
   closepath          % cleanly close path!
   setgray            % grey -> __
   fill               % colour in region
} def
% Script -- usually generated
0 100 100 box
0.4 200 200 box
0.6 300 300 box
0 setgray
showpage
```



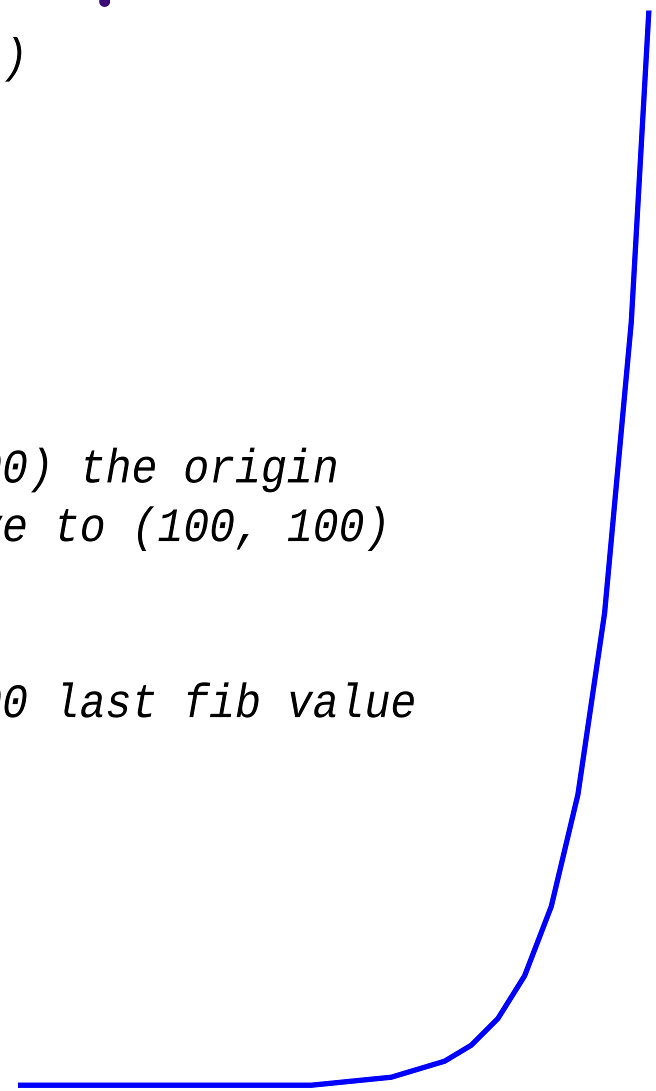
# Graphics state and coordinate operators

| | | | |
|---|---|---|---|
| num | `setlinewidth` | - | set line width |
| num | `setgray` | - | set colour to gray value (0 = black; 1 = white) |
| $s_x$ $s_y$ | `scale` | - | scale use space by $s_x$ and $s_y$ |
| angle | `rotate` | - | rotate user space by *angle* degrees |
| $t_x$ $t_y$ | `translate` | - | translate user space by $(t_x, t_y)$ |
| - | `matrix` | matrix | create identity matrix |
| matrix | `currentmatrix` | matrix | fill *matrix* with CTM |
| matrix | `setmatrix` | - | replace CTM by *matrix* |
| - | `gsave` | - | save graphics state |
| - | `grestore` | - | restore graphics state |

*gsave saves the current path, gray value, line width and user coordinate system*

# A Fibonacci Graph

```
/fibInc {                     % m n -> n (m+n)
   exch                       % m n -> n m
   1 index                    % n m -> n m n
   add
} def
/x 0 def /y 0 def /dx 10 def
newpath
100 100 translate             % make (100, 100) the origin
x y moveto                    % i.e., relative to (100, 100)
0 1 25 {
   /x x dx add def            % increment x
   dup /y exch 100 idiv def % set y to 1/100 last fib value
   x y lineto                 % draw segment
   fibInc
} repeat
2 setlinewidth
stroke
showpage
```



# Numbers and Strings

Numbers and other objects must be converted to strings before they can be printed:

| | | | |
|---|---|---|---|
| int | `string` | string | create string of capacity *int* |
| any string | `cvs` | substring | convert to string |

# Factorial

```
/LM 100 def             % left margin
/FS 18 def              % font size
/sBuf 20 string def     % string buffer of length 20
/fact {                 % n -> n!
   dup 1 lt             % -> n bool
   { pop 1 }            % 0 -> 1
   {
      dup               % n -> n n
      1                 % -> n n 1
      sub               % -> n (n-1)
      fact              % -> n (n-1)!  NB: recursive lookup
      mul               % n!
   }
   ifelse
} def
/showInt {              % n -> __
   sBuf cvs show        % convert an integer to a string and show it
} def
```

# Factorial ...

```
/showFact {                        % n -> __
   dup showInt                     % show n
   (! = ) show                     % ! =
   fact showInt                    % show n!
} def
/newline {                         % __ -> __
   currentpoint exch pop           % get current y
   FS 2 add sub                    % subtract offset
   LM exch moveto                  % move to new x y
} def

/Times-Roman findfont FS scalefont setfont
LM 600 moveto
0 1 20 { showFact newline } for % do from 0 to 20
showpage
```

| |
|---|
| 0! = 1 |
| 1! = 1 |
| 2! = 2 |
| 3! = 6 |
| 4! = 24 |
| 5! = 120 |
| 6! = 720 |
| 7! = 5040 |
| 8! = 40320 |
| 9! = 362880 |
| 10! = 3628800 |
| 11! = 39916800 |
| 12! = 479001600 |
| 13! = 6.22702e+09 |
| 14! = 8.71783e+10 |
| 15! = 1.30767e+12 |
| 16! = 2.09228e+13 |
| 17! = 3.55687e+14 |
| 18! = 6.40237e+15 |
| 19! = 1.21645e+17 |
| 20! = 2.4329e+18 |

# Boolean, control and string operators

| | | | |
|---|---|---|---|
| $any_1$ $any_2$ | `eq` | bool | test equal |
| $any_1$ $any_2$ | `ne` | bool | test not equal |
| $any_1$ $any_2$ | `ge` | bool | test greater or equal |
| - | `true` | true | push boolean value *true* |
| - | `false` | bool | test equal |
| bool proc | `if` | - | execute *proc* if *bool* is true |
| bool $proc_1$ $proc_2$ | `ifelse` | - | execute $proc_1$ if *bool* is true else $proc_2$ |
| init incr limit proc | `for` | - | execute *proc* with values *init* to *limit* by steps of *incr* |
| int proc | `repeat` | - | execute *proc* *int* times |
| string | `length` | int | number of elements in *string* |
| string index | `get` | int | get element at position *index* |
| string index int | `put` | - | put *int* into *string* at position *index* |
| string proc | `forall` | - | execute *proc* for each element of *string* |

# A simple formatter

```
/LM 100 def                  % left margin
/RM 250 def                  % right margin
/FS 18 def                   % font size
/showStr {                   % string -> __
   dup stringwidth pop       % get (just) string's width
   currentpoint pop          % current x position
   add                       % where printing would bring us
   RM gt { newline } if      % newline if this would overflow RM
   show
} def
/newline {                   % __ -> __
   currentpoint exch pop     % get current y
   FS 2 add sub              % subtract offset
   LM exch moveto            % move to new x y
} def
/format { { showStr ( ) show } forall } def    % array -> __
/Times-Roman findfont FS scalefont setfont
LM 600 moveto
```

# A simple formatter ...

```
[ (Now) (is) (the) (time) (for) (all) (good) (men) (to)
(come) (to) (the) (aid) (of) (the) (party.) ] format
showpage
```

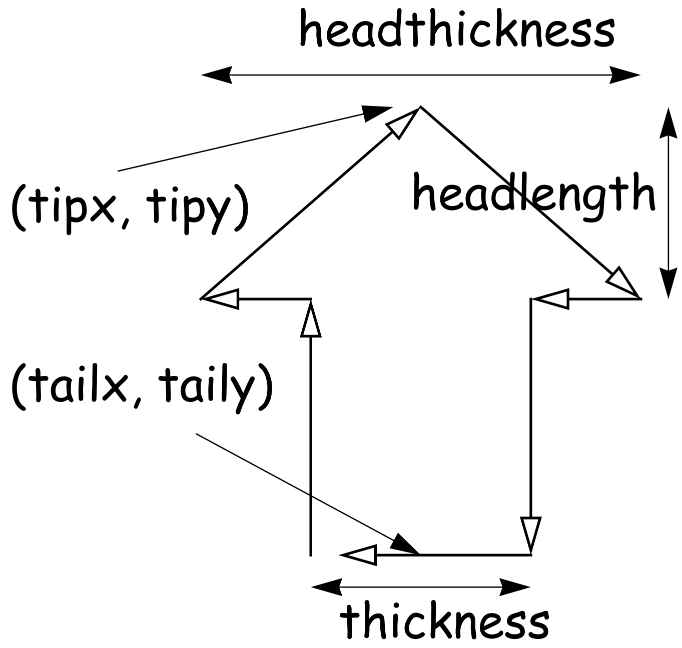
| |
|---|
| Now is the time for<br>all good men to<br>come to the aid of<br>the party. |

# Array and dictionary operators

| | | | |
|---|---|---|---|
| - | `[` | mark | start array construction |
| mark $obj_0$ … $obj_{n-1}$ | `]` | array | end array construction |
| int | `array` | array | create array of length *n* |
| array | `length` | int | number of elements in array |
| array index | `get` | any | get element at *index* position |
| array index any | `put` | - | put element at *index* position |
| array proc | `forall` | - | execute *proc* for each *array* element |
| int | `dict` | dict | create dictionary of capacity *int* |
| dict | `length` | int | number of key-value pairs |
| dict | `maxlength` | int | capacity |
| dict | `begin` | - | push *dict* on dict stack |
| - | `end` | - | pop dict stack |

# Using Dictionaries — Arrowheads

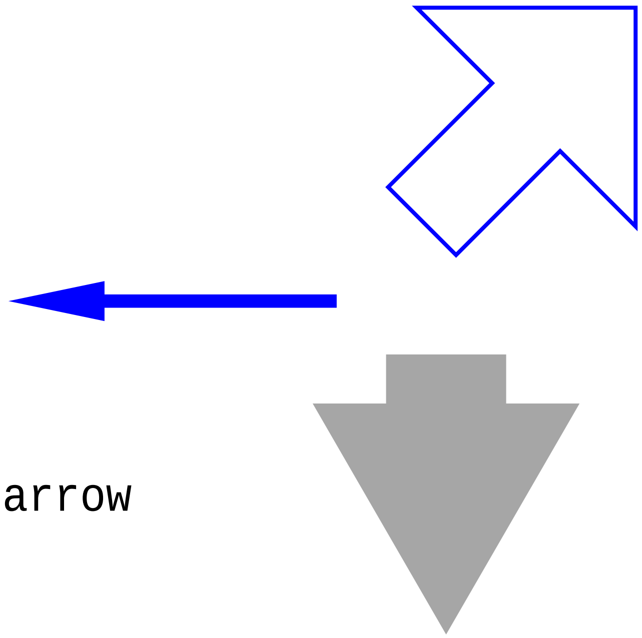
```
/arrowdict 14 dict def                  % make a new dictionary
arrowdict begin
  /mtrx matrix def                      % allocate space for a matrix
end
/arrow {
  arrowdict begin % open the dictionary
    /headlength exch def % grab args
    /halfheadthickness exch 2 div def
    /halfthickness exch 2 div def
    /tipy exch def
    /tipx exch def
    /taily exch def
    /tailx exch def
    /dx tipx tailx sub def
    /dy tipy taily sub def
    /arrowlength dx dx mul dy dy mul add sqrt def
    /angle dy dx atan def
    /base arrowlength headlength sub def
```



```
    /savematrix mtrx currentmatrix def % save the coordinate system
    tailx taily translate              % translate to start of arrow
    angle rotate                       % rotate coordinates
    0 halfthickness neg moveto         % draw as if starting from (0,0)
    base halfthickness neg lineto
    base halfheadthickness neg lineto
    arrowlength 0 lineto
    base halfheadthickness lineto
    base halfthickness lineto
    0 halfthickness lineto
    closepath
    savematrix setmatrix               % restore coordinate system
  end
} def
```

# Instantiating Arrows

```
newpath
   318 340 72 340 10 30 72 arrow
fill
newpath
   382 400 542 560 72 232 116 arrow
3 setlinewidth stroke
newpath
   400 300 400 90 90 200 200 3 sqrt mul 2 div arrow
.65 setgray fill
showpage
```



# Encapsulated PostScript

EPSF is a standard format for importing and exporting PostScript files between applications.

```
%!PS-Adobe-3.0 EPSF-3.0
%%BoundingBox: 90 490 200 520
/Times-Roman findfont
        18 scalefont
        setfont
100 500 moveto
(Hello world) show
showpage
```

# What you should know!

- What kinds of stacks does PostScript manage?
- When does PostScript push values on the operand stack?
- What is a path, and how can it be displayed?
- How do you manipulate the coordinate system?
- Why would you define your own dictionaries?
- How do you compute a bounding box for your PostScript graphic?

# Can you answer these questions?

- *How would you program this graphic?* 

- *When should you use translate instead of moveto?*
- *How could you use dictionaries to simulate object-oriented programming?*

# 3. Functional Programming

**Overview**

- Functional vs. Imperative Programming
- Referential Transparency
- Recursion
- Pattern Matching
- Higher Order Functions
- Lazy Lists

# References


- Paul Hudak, "Conception, Evolution, and Application of Functional Programming Languages," ACM Computing Surveys 21/3, pp 359-411.
- Paul Hudak and Joseph H. Fasel, "A Gentle Introduction to Haskell," ACM SIGPLAN Notices, vol. 27, no. 5, May 1992, pp. T1-T53.
- Simon Peyton Jones and John Hughes [editors], *Report on the Programming Language Haskell 98 A Non-strict, Purely Functional Language*, February 1999
  - ☞ www.haskell.org

# A Bit of History

| | |
|---|---|
| *Lambda Calculus* (Church, 1932-33) | formal model of computation |
| *Lisp* (McCarthy, 1960) | symbolic computations with lists |
| *APL* (Iverson, 1962) | algebraic programming with arrays |
| *ISWIM* (Landin, 1966) | ***let*** and ***where*** clauses |
| | equational reasoning; birth of "pure" functional programming ... |

# A Bit of History

| | |
|---|---|
| *ML* (Edinburgh, 1979) | originally meta language for theorem proving |
| *SASL, KRC, Miranda* (Turner, 1976-85) | lazy evaluation |
| *Haskell* (Hudak, Wadler, et al., 1988) | "Grand Unification" of functional languages ... |

# Programming without State

**Imperative style:**

```
n := x;
a := 1;
while n>0 do
begin a:= a*n;
  n := n-1;
end;
```

**Declarative (functional) style:**

```
fac n =
      if    n == 0
      then  1
      else  n * fac (n-1)
```

*Programs in pure functional languages have <u>no explicit state</u>.*
*Programs are constructed entirely by composing expressions.*

# Pure Functional Programming Languages

**Imperative Programming:**

☞ Program = Algorithms + Data

**Functional Programming:**

☞ Program = Functions $\circ$ Functions

**What is a Program?**

A program (computation) is a transformation from input data to output data.

# Key features of pure functional languages

1. *All programs* and procedures are *functions*
2. There are *no variables* or *assignments* — only input parameters
3. There are *no loops* — only recursive functions
4. The value of a function *depends only on* the values of its *parameters*
5. Functions are *first-class values*

# Haskell

*Haskell is a general purpose, purely functional programming language incorporating many recent innovations in programming language design. Haskell provides higher-order functions, non-strict semantics, static polymorphic typing, user-defined algebraic datatypes, pattern-matching, list comprehensions, a module system, a monadic I/O system, and a rich set of primitive datatypes, including lists, arrays, arbitrary and fixed precision integers, and floating-point numbers. Haskell is both the culmination and solidification of many years of research on lazy functional languages.*

*— The Haskell 98 report*

# Referential Transparency

A function has the property of *referential transparency* if its value depends only on the values of its parameters.

✎ *Does* `f(x)+f(x)` *equal* `2*f(x)`*? In C? In Haskell?*

Referential transparency means that "*equals can be replaced by equals*".

In a pure functional language, all functions are referentially transparent, and therefore *always yield the same result* no matter how often they are called.

# Evaluation of Expressions

Expressions can be (formally) evaluated by substituting arguments for formal parameters in function bodies:

```
fac 4   ⇨ if 4 == 0 then 1 else 4 * fac (4-1)
        ⇨ 4 * fac (4-1)
        ⇨ 4 * (if (4-1) == 0 then 1 else (4-1) * fac (4-1-1))
        ⇨ 4 * (if 3 == 0 then 1 else (4-1) * fac (4-1-1))
        ⇨ 4 * ((4-1) * fac (4-1-1))
        ⇨ 4 * ((4-1) * (if (4-1-1) == 0 then 1 else (4-1-1) * ...))
        ⇨ ...
        ⇨ 4 * ((4-1) * ((4-1-1) * ((4-1-1-1) * 1)))
        ⇨ ...
        ⇨ 24
```

*Of course, real functional languages are not implemented by syntactic substitution ...*

# Tail Recursion

Recursive functions can be less efficient than loops because of the *high cost of procedure calls* on most hardware.

A *tail recursive function* calls itself *only* as its last operation, so the recursive call can be *optimized away* by a modern compiler since it needs only a single run-time stack frame:

```
fact 5  →  fact 5|fact 4  →  fact 5|fact 4|fact 3

     sfac 5  →  sfac 4  →  sfac 3
```

...

# Tail Recursion ...

A recursive function can be *converted* to a tail-recursive one by representing partial computations as *explicit function parameters:*

```
sfac s n = if    n == 0
           then  s
           else  sfac (s*n) (n-1)

sfac 1 4 ➪  sfac (1*4) (4-1)
         ➪  sfac 4 3
         ➪  sfac (4*3) (3-1)
         ➪  sfac 12 2
         ➪  sfac (12*2) (2-1)
         ➪  sfac 24 1
         ➪  ... ➪ 24
```

# Equational Reasoning

**Theorem:**

For all $n \geq 0$, `fac n = sfac 1 n`

**Proof of theorem:**

$n = 0$: `fac 0 = 1 = sfac 1 0`

$n > 0$: Suppose

```
fac (n-1)  = sfac 1 (n-1)
fac n      = n * fac (n-1)     — by def
           = n * sfac 1 (n-1)
           = sfac n (n-1)      — by lemma
           = sfac 1 n          — by def
```

...

# Equational Reasoning ...

**Lemma:**

For all $n \geq 0$, `sfac s n = s * sfac 1 n`

**Proof of lemma:**

$n = 0$: `sfac s 0 = s = s * sfac 1 0`

$n > 0$: Suppose:

```
sfac s (n-1) = s * sfac 1 (n-1)
sfac s n     = sfac (s*n) (n-1)
             = s * n * sfac 1 (n-1)
             = s * sfac n (n-1)
             = s * sfac 1 n
```

# Pattern Matching

Haskell support multiple styles for specifying case-based function definitions:

**Patterns:**

```
fac' 0  = 1
fac' n  = n * fac' (n-1)

-- or: fac' (n+1) = (n+1) * fac' n
```

**Guards:**

```
fac'' n | n == 0 = 1
        | n >= 1 = n * fac'' (n-1)
```

# Lists

Lists are *pairs* of *elements* and *lists* of elements:

- `[ ]` — stands for the empty list
- `x:xs` — stands for the list with `x` as the head and `xs` as the rest of the list
- `[1,2,3]` — is syntactic sugar for `1:2:3:[ ]`
- `[1..n]` — stands for `[1,2,3, ... n]`

# Using Lists

Lists can be *deconstructed* using *patterns:*

```
head (x:_) = x

len [ ]       = 0
len (x:xs)    = 1 + len xs

prod [ ]      = 1
prod (x:xs)   = x * prod xs

fac''' n      = prod [1..n]
```

# Higher Order Functions

Higher-order functions treat other functions as *first-class values* that can be composed to produce new functions.

```
map f [ ]     = [ ]
map f (x:xs) = f x : map f xs
```

```
map fac [1..5]
        ➪  [1, 2, 6, 24, 120]
```

NB: `map fac` is a new function that can be applied to lists:

```
mfac = map fac
mfac [1..3]
        ➪  [1, 2, 6]
```

# Anonymous functions

Anonymous functions can be written as “lambda abstractions”.

The function `(\x -> x * x)` behaves exactly like `sqr`:

```
sqr x = x * x

sqr 10               ➪ 100
(\x -> x * x) 10     ➪ 100
```

Anonymous functions are first-class values:

```
map (\x -> x * x) [1..10]
      ➪ [1, 4, 9, 16, 25, 36, 49, 64, 81, 100]
```

# Curried functions

A <u>Curried function</u> [named after the logician H.B. Curry] *takes its arguments one at a time*, allowing it to be treated as a higher-order function.

```
plus x y    = x + y         -- curried addition
plus 1 2    ⇨ 3

inc      = plus 1           -- bind first argument to 1
inc 2    ⇨ 3

fac = sfac 1                -- binds first argument of
      where sfac s n        -- a curried factorial
            | n == 0 = s
            | n >= 1= sfac (s*n) (n-1)
```

# Understanding Curried functions

```
plus x y = x + y
```

*is the same as:*

```
plus x = \y -> x+y
```

In other words, plus is *a function of one argument* that *returns a function* as its result.

```
plus 5 6
```

*is the same as:*

```
(plus 5) 6
```

In other words, we invoke (plus 5), obtaining a function,

```
\y -> 5 + y
```

which we then pass the argument 6, yielding 11.

# Currying

The following (pre-defined) function takes a binary function as an argument and turns it into a curried function:

```
curry f a b  = f (a, b)

plus(x,y)  = x + y              -- not curried!
inc        = (curry plus) 1

sfac(s, n) = if    n == 0       -- not curried
             then  s
             else  sfac (s*n, n-1)

fac = (curry sfac) 1            -- bind first argument
```

# Multiple Recursion

*Naive* recursion may result in *unnecessary* recalculations:

```
fib 1       = 1
fib 2       = 1
fib (n+2)   = fib n + fib (n+1)
```

Efficiency can be regained by *explicitly passing* calculated values:

```
fib' 1    = 1
fib' n    = a         where (a,_) = fibPair n
fibPair 1       = (1,0)
fibPair (n+2)   = (a+b,a)
      where (a,b) = fibPair (n+1)
```

✎ *How would you write a tail-recursive Fibonacci function?*

# Lazy Evaluation

"Lazy", or "normal-order" evaluation only evaluates expressions *when they are actually needed*. Clever implementation techniques (Wadsworth, 1971) allow replicated expressions to be shared, and thus avoid needless recalculations.

So:

```
sqr n = n * n
sqr (2+5) ⇨ (2+5) * (2+5) ⇨ 7 * 7 ⇨ 49
```

Lazy evaluation allows some functions to be evaluated even if they are passed incorrect or non-terminating arguments:

```
ifTrue True x y  = x
ifTrue False x y = y
ifTrue True 1 (5/0) ⇨ 1
```

# Lazy Lists

Lazy lists are *infinite data structures* whose values are generated by need:

```
from n = n : from (n+1)

from 10 ➪ [10,11,12,13,14,15,16,17,....

take 0 _            = [ ]
take _ [ ]          = [ ]
take (n+1) (x:xs)  = x : take n xs

take 5 (from 10) ➪ [10, 11, 12, 13, 14]
```

*NB: The lazy list (from n) has the special syntax: [n..]*

# Programming lazy lists

Many sequences are naturally implemented as lazy lists.
*Note the top-down, declarative style:*

```
fibs = 1 : 1 : fibsFollowing 1 1
     where fibsFollowing a b =
       (a+b) : fibsFollowing b (a+b)
```

```
take 10 fibs
      ➪ [ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 ]
```

✎ *How would you re-write fibs so that (a+b) only appears once?*

# Declarative Programming Style

```
primes = primesFrom 2
primesFrom n = p : primesFrom (p+1)
                 where p = nextPrime n
nextPrime n
  | isPrime n  = n
  | otherwise  = nextPrime (n+1)
isPrime 2      = True
isPrime n      = notDivisible primes n
notDivisible (k:ps) n
  | (k*k) > n       = True
  | (mod n k) == 0 = False
  | otherwise       = notDivisible ps n
```

**take 100 primes** ➪ [ 2, 3, 5, 7, 11, 13, ... 523, 541 ]

# What you should know!

- What is referential transparency? Why is it important?
- When is a function tail recursive? Why is this useful?
- What is a higher-order function? An anonymous function?
- What are curried functions? Why are they useful?
- How can you avoid recalculating values in a multiply recursive function?
- What is lazy evaluation?
- What are lazy lists?

# Can you answer these questions?

- Why don't pure functional languages provide loop constructs?
- When would you use patterns rather than guards to specify functions?
- Can you build a list that contains both numbers and functions?
- How would you simplify `fibs` so that `(a+b)` is only called once?
- What kinds of applications are well-suited to functional programming?

# 4. Type Systems

**Overview**

- What is a Type?
- Static vs. Dynamic Typing
- Kinds of Types
- Polymorphic Types
- Overloading
- User Data Types

# References


- Paul Hudak, "Conception, Evolution, and Application of Functional Programming Languages," ACM Computing Surveys 21/3, Sept. 1989, pp 359-411.
- L. Cardelli and P. Wegner, "On Understanding Types, Data Abstraction, and Polymorphism,'"ACM Computing Surveys, 17/4, Dec. 1985, pp. 471-522.
- D. Watt, *Programming Language Concepts and Paradigms*, Prentice Hall, 1990

# What is a Type?

**Type errors:**

```
? 5 + [ ]
ERROR: Type error in application
*** expression : 5 + [ ]
*** term : 5
*** type : Int
*** does not match : [a]
```

**A type is a set of values?**

- int = { ... -2, -1, 0, 1, 2, 3, ... }
- bool = { True, False }
- Point = { [x=0,y=0], [x=1,y=0], [x=0,y=1] ... }

# What is a Type?

**A type is a partial specification of behaviour?**

- `n,m:int` ⇒ `n+m` is valid, but `not(n)` is an error
- `n:int` ⇒ `n := 1`is valid, but `n := “hello world”` is an error

*What kinds of specifications are interesting? Useful?*

# Static and Dynamic Types

*Values* have *static types* defined by the programming language.

*Variables* and *expressions* have *dynamic types* determined by the values they assume at run-time.

*declared*, static type is Applet

static type of *value* is GameApplet

```
Applet myApplet = new GameApplet();
```

*actual* dynamic type is GameApplet

# Static and Dynamic Typing

A language is *statically typed* if it is always possible to determine the (static) type of an expression *based on the program text alone.*

A language is *strongly typed* if it is possible to ensure that every expression is *type consistent* based on the program text alone.

A language is *dynamically typed* if *only values have fixed type*. Variables and parameters may take on different types at run-time, and must be checked immediately before they are used.

Type consistency may be assured by (i) *compile-time type-checking*, (ii) *type inference*, or (iii) *dynamic type-checking*.

# Kinds of Types

**All programming languages provide some set of built-in types.**

- *Primitive types:* booleans, integers, floats, chars ...
- *Composite types:* functions, lists, tuples ...

Most strongly-typed modern languages provide for additional user-defined types.

- *User-defined types:* enumerations, recursive types, generic types, objects ...

# Type Completeness

**The Type Completeness Principle:**

> *No operation should be arbitrarily restricted in the types of values involved.* — *Watt*

First-class values can be *evaluated*, *passed* as arguments and used as *components* of composite values.

Functional languages attempt to make *no class distinctions*, whereas imperative languages typically treat functions (at best) as *second-class* values.

# Function Types

Function types allow one to *deduce* the types of expressions without the need to evaluate them:

```
fact :: Int -> Int
42 :: Int                 ⇒   fact 42 :: Int
```

**Curried types:**

```
Int -> Int -> Int         ≡   Int -> (Int -> Int)
```

and

```
plus 5 6                  ≡   ((plus 5) 6).
```

so:

```
plus::Int->Int->Int       ⇒   plus 5::Int->Int
```

# List Types

**List Types**

A list of values of type `a` has the type `[a]`:

*`[ 1 ] :: [ Int ]`*

*NB: All of the elements in a list must be of the same type!*

`['a', 2, False]`-- *this is illegal! can't be typed!*

# Tuple Types

**Tuple Types**

If the expressions `x1`, `x2`, ..., `xn` have types `t1`, `t2`, ..., `tn` respectively, then the tuple `(x1, x2, ..., xn)`has the type `(t1, t2, ..., tn)`:

```
(1, [2], 3) :: (Int, [Int], Int)
('a', False) :: (Char, Bool)
((1,2),(3,4)) :: ((Int, Int), (Int, Int))
```

The unit type is written `()` and has a single element which is also written as `()`.

# Monomorphism

Languages like Pascal have *monomorphic type systems:* every constant, variable, parameter and function result has a *unique* type.

- *good* for *type-checking*
- *bad* for writing *generic* code
  - ☞ it is impossible in Pascal to write a generic sort procedure

# Polymorphism

A <u>polymorphic function</u> accepts *arguments of different types:*

```
length          :: [a] -> Int
length [ ]      = 0
length (x:xs)   = 1 + length xs

map             :: (a -> b) -> [a] -> [b]
map f [ ]       = [ ]
map f (x:xs)    = f x : map f xs

(.)             :: (b -> c) -> (a -> b) -> (a -> c)
(f . g) x       = f (g x)
```

# Composing polymorphic types

We can *deduce* the types of expressions using polymorphic functions by simply *binding type variables to concrete types.*

Consider:

```
length        :: [a] -> Int
map           :: (a -> b) -> [a] -> [b]
```

Then:

```
map length                          :: [[a]] -> [Int]
[ "Hello", "World" ]                :: [[Char]]
map length [ "Hello", "World" ]     :: [Int]
```

# Polymorphic Type Inference

Hindley-Milner Type Inference provides an effective algorithm for automatically determining the types of polymorphic functions.

```
map    f    [ ]     =  [ ]
map    f    (x:xs)  =  f x : map f xs

map  ::  X       ->  Y      ->  Z
map  ::  (a -> b) -> [ c ]  ->  [ d ]
map  ::  (a -> b) -> [ a ]  ->  [ b ]
```

The corresponding type system is used in many modern functional languages, including ML and Haskell.

# Type Specialization

A polymorphic function may be explicitly assigned a *more specific* type:

```
idInt :: Int -> Int
idInt x = x
```

Note that the :t command can be used to find the type of a particular expression that is inferred by Haskell:

```
? :t \x -> [x]
➪ \x -> [x] :: a -> [a]

? :t (\x -> [x]) :: Char -> String
➪ \x -> [x] :: Char -> String
```

# Kinds of Polymorphism

**Polymorphism:**

- Universal:
  - *Parametric: polymorphic map function in Haskell; nil pointer type in Pascal*
  - *Inclusion: subtyping — graphic objects*
- Ad Hoc:
  - *Overloading: + applies to both integers and reals*
  - *Coercion: integer values can be used where reals are expected and v.v.*

# Coercion vs overloading

Coercion or overloading — how does one distinguish?

```
3 + 4
3.0 + 4
3 + 4.0
3.0 + 4.0
```

✎ *Are there several overloaded + functions, or just one, with values automatically coerced?*

# Overloading

Overloaded operators are introduced by means of *type classes*:

```
class Eq a where
  (==), (/=) :: a -> a -> Bool
  x /= y  = not (x == y)
```

A type class must be *instantiated* to be used:

```
instance Eq Bool where
  True == True               = True
  False == False             = True
  _ == _                     = False
```

# Instantiating overloaded operators

For each overloaded instance a separate definition must be given ...

```
instance Eq Int where (==)  = primEqInt
instance Eq Char where c == d   = ord c == ord d
instance (Eq a, Eq b) => Eq (a,b) where
  (x,y) == (u,v)                = x==u && y==v
instance Eq a => Eq [a] where
  [ ] == [ ]                    = True
  [ ] == (y:ys)                 = False
  (x:xs) == [ ]                 = False
  (x:xs) == (y:ys)              = x==y && xs==ys
```

# User Data Types

New data types can be introduced by specifying (i) a *datatype name*, (ii) a set of *parameter types*, and (iii) a set of *constructors* for elements of the type:

```
data DatatypeName a1 ... an = constr1 | ... | constrm
```

where the constructors may be either:

1. *Named* constructors:
```
Name type1 ... typek
```
2. *Binary* constructors (i.e., starting with ":"):
```
type1 CONOP type2
```

# Enumeration types

User data types that do not hold any data can model enumerations:

```
data Day = Sun | Mon | Tue | Wed | Thu | Fri | Sat
```

Functions over user data types must *deconstruct* the arguments, with one case for each constructor:

```
whatShallIDo Sun   = "relax"
whatShallIDo Sat   = "go shopping"
whatShallIDo _     = "guess I'll have to go to work"
```

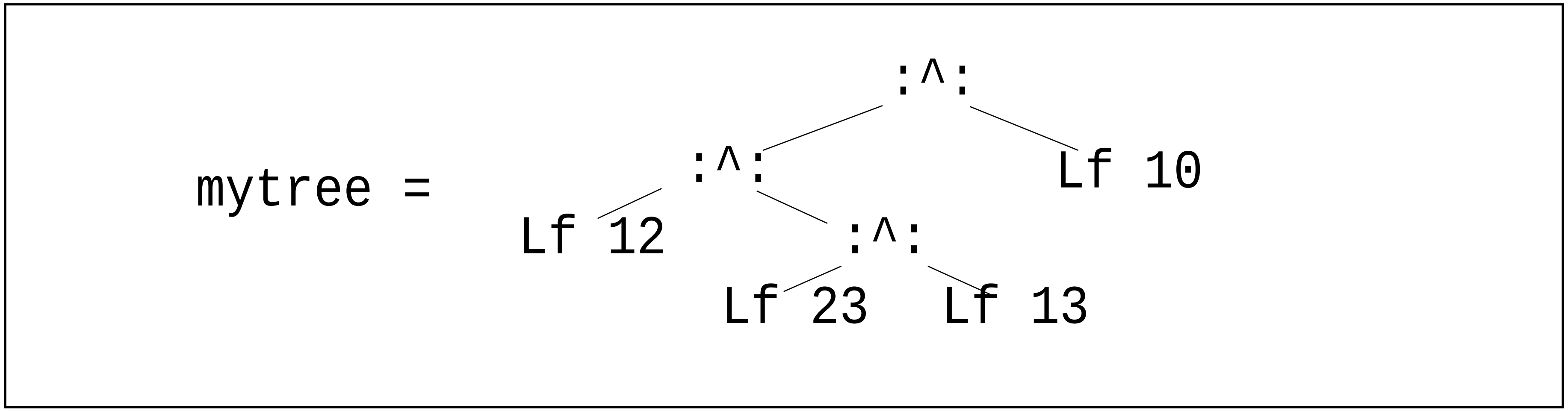
# Union types

```
data Temp = Centigrade Float | Fahrenheit Float

freezing :: Temp -> Bool
freezing (Centigrade temp)= temp <= 0.0
freezing (Fahrenheit temp)= temp <= 32.0
```

# Recursive Data Types

A recursive data type provides constructors over the type itself:

```
data Tree a = Lf a | Tree a :^: Tree a

mytree = (Lf 12 :^: (Lf 23 :^: Lf 13)) :^: Lf 10
```




```
? :t mytree ⇨ mytree :: Tree Int
```

# Using recursive data types

```
leaves, leaves' :: Tree a -> [a]
leaves (Lf l)       = [l]
leaves (l :^: r)    = leaves l ++ leaves r

leaves' t = leavesAcc t [ ]
  where leavesAcc (Lf l) = (l:)
        leavesAcc (l :^: r) = leavesAcc l . leavesAcc r
```

- ✎ *What do these functions do?*
- ✎ *Which function should be more efficient? Why?*
- ✎ *What is (l:) and what does it do?*

# Equality for Data Types

*Why not automatically provide equality for all types of values?*

**User data types:**

```
data Set a = Set [a]
instance Eq a => Eq (Set a) where
  Set xs == Set ys = xs `subset` ys && ys `subset` xs
    where xs `subset` ys = all (`elem` ys) xs
```

*NB: all ('elem' ys) xs tests that every x in xs is an element of ys*

# Equality for Functions

**Functions:**

```
? (1==) == (\x->1==x)
ERROR: Cannot derive instance in expression
*** Expression        : (==) d148 ((==) {dict} 1) (\x->(==) {dict} 1 x)
*** Required instance : Eq (Int -> Bool)
```

Determining equality of functions is *undecidable* in general!

# What you should know!

- How are the types of functions, lists and tuples specified?
- How can the type of an expression be inferred without evaluating it?
- What is a polymorphic function?
- How can the type of a polymorphic function be inferred?
- How does overloading differ from parametric polymorphism?
- How would you define == for tuples of length 3?
- How can you define your own data types?
- Why isn't == pre-defined for all types?

# Can you answer these questions?

- *Can any set of values be considered a type?*
- *Why does Haskell sometimes fail to infer the type of an expression?*
- *What is the type of the predefined function* `all`*? How would you implement it?*

# 5. An application of Functional Programming

**Overview**

- Huffmann encoding
  - ☞ variable length encoding based on character frequency
- Architecture of a functional Huffmann encoder
- How to use recursion correctly ☞ *ensuring termination*
- Representing and manipulating trees
- Encoding trees as text; parsing stored trees
- Continuation-style IO
- "It doesn't always pay to be lazy!" — forcing eager evaluation

# Reference


- H. Abelson, G. Sussman and J.Sussman, *Structure and Interpretation of Computer Programs*, MIT electrical engineering and computer science series., McGraw-Hill, 1991.

# Encoding ASCII

```
"I am what I am."
```

Naive encoding requires *at least 4 bits* to encode 9 different characters:

| | |
|---|---|
| " | 0000 |
| I | 0001 |
| (blank) | 0010 |
| a | 0011 |
| m | 0100 |
| w | 0101 |
| h | 0110 |
| t | 0111 |
| . | 1000 |

16 characters x 4 bits/character = 64 bits

```
0000 0001 0010 0011 0100 0010 0101
0110 0011 0111 0010 0001 0010 0011
0100 0000
```

# Huffmann encoding

Huffmann encoding assigns *fewer* bits to more *frequently used* characters.

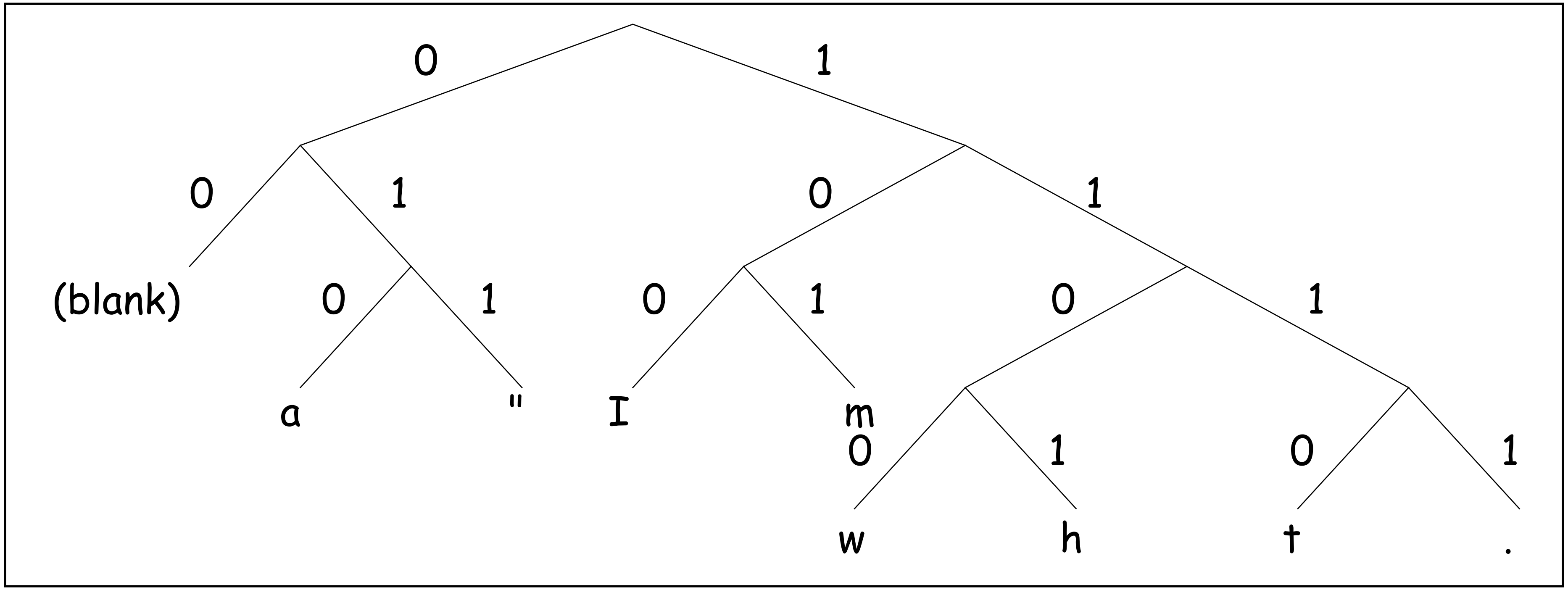
| *char* | *frequency* | *encoding* |
|---|---|---|
| (blank) | 4 | 00 |
| a | 3 | 010 |
| " | 2 | 011 |
| I | 2 | 100 |
| m | 2 | 101 |
| w | 1 | 1100 |
| h | 1 | 1101 |
| t | 1 | 1110 |
| . | 1 | 1111 |

$4 \times 2 + 9 \times 3 + 4 \times 4 = 51$ bits

```
011 100 00 010 101 00 1100
1101 010 1110 00 100 00 010
101 011
```

# Huffmann decoding

A Huffmann encoded text can be decoded by using the bits to *walk down the encoding tree* and outputting the characters at the leaves:




`011 100 00 010 101 00 1100 1101 010 1110 00`

⇨ `“I am what ...`

# Generating optimal trees

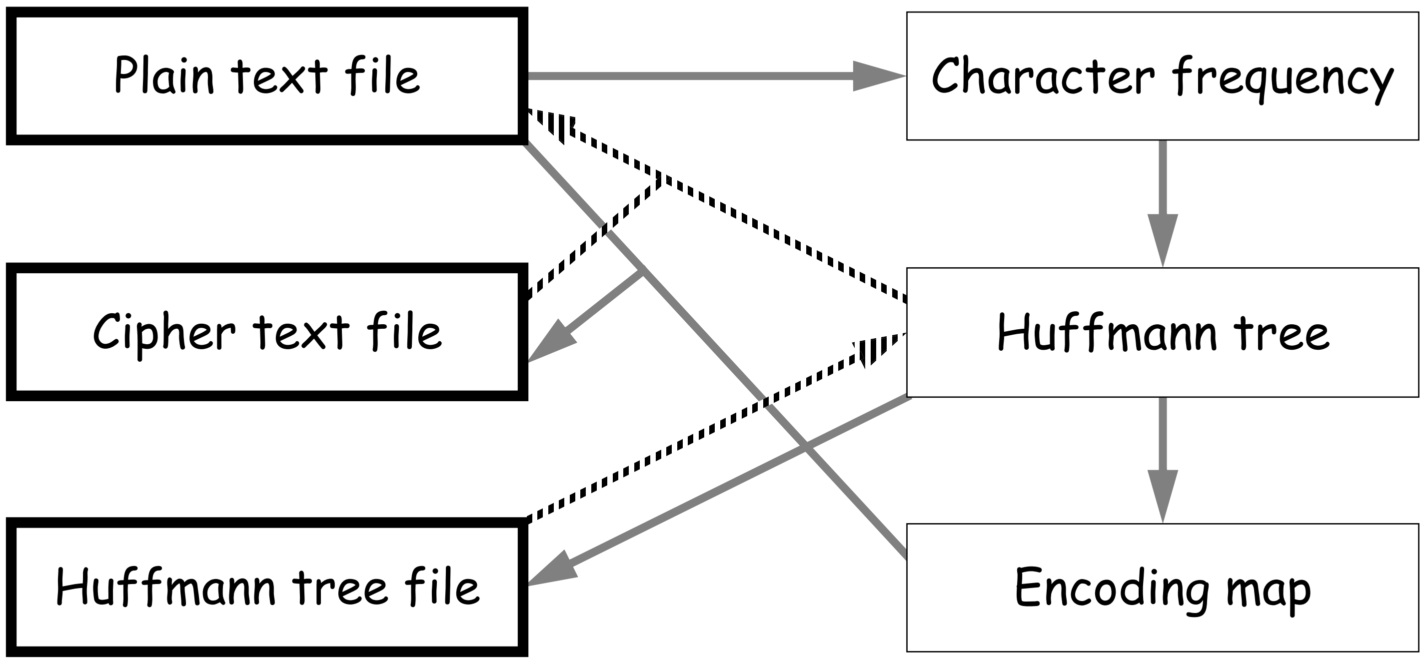
Huffmann's algorithm generates the *optimal* encoding/ decoding tree by *recursively merging* the two "smallest" (by weight) subtrees:

- ➪ blank$_4$ a$_3$ I$_2$ m$_2$ w$_1$ h$_1$ t$_1$ .$_1$
- ➪ blank$_4$ a$_3$ I$_2$ m$_2$ w$_1$ h$_1$ (t .)$_2$
- ➪ blank$_4$ a$_3$ I$_2$ m$_2$ (w h)$_2$ (t .)$_2$
- ➪ blank$_4$ a$_3$ I$_2$ m$_2$ ((w h) (t .))$_4$
- ➪ blank$_4$ a$_3$ (I m)$_4$ ((w h) (t .))$_4$
- ➪ (blank a)$_7$ (I m)$_4$ ((w h) (t .))$_4$
- ➪ (blank a)$_7$ ((I m) ((w h) (t .)))$_8$
- ➪ ((blank a) ((I m) ((w h) (t .))))$_{15}$

✎ *Write a program to Huffmann encode and decode text files.*

# Architecture

At the coarsest granularity, we need three components to encode and decode files:



# A Simple testing framework

A test consists of a *single named test case*, or a *suite* of tests:

```
data Test name test =
      Test name test
    | Test name test :+: Test name test
      deriving Show
```

We return only the names of tests that fail:

```
dotest (Test name test) =
  if (test ())
  then ""
  else name ++ " FAILED\n"
dotest (t1 :+: t2) =
  (dotest t1) ++ (dotest t2)
```

# Testing

```
assert test =
  let result = dotest test
  in
    if length(result) > 0
    then putStr result
    else putStr "PASSED all tests"

tests =
      Test "test1" (\x -> 1 == 1)
  :+: Test "test2" (\x -> 2 == 2)
```

**`assert allTests`**

➪ `PASSED all tests`

# Frequency Counting

*We represent frequencies as lists of pairs of Chars and Ints:*

```
type CharCount    = (Char,Int)
```

*Compute a [CharCount] for a given String*

```
freqCount :: String -> [CharCount]
freqCount ""      = []
freqCount (c:s)   = incCount c (freqCount s)
```

*Increment the [CharCount] for a given Char*

```
incCount :: Char -> [CharCount] -> [CharCount]
incCount c []     = [(c,1)]
incCount c ((c1,n):ccList)
  | c == c1       = (c1,n+1):ccList
  | otherwise     = (c1,n):(incCount c ccList)
```

# How to use recursion correctly!

In order to ensure that a recursive function will terminate:

1. Carefully *establish the base cases:*

```
freqCount ""      = []
```

☞ base case is *an empty string*

2. Ensure that every recursive invocation *reduces some measure of size*, and therefore will eventually reach a base case:

```
freqCount (c:s)  = incCount c (freqCount s)
```

☞ recursive call reduces *length of argument string* ⇒ will reach base case

# Freqcount tests

```
iam = "\"I am what I am.\""
freqCount iam
  ➪ [('"',2), ('.',1), ('m',2), ('a',3), (' ',4),
    ('I',2), ('t',1), ('h',1), ('w',1)]

testFreqCount = let result = freqCount iam in
      Test "freqCount length"
            (\x -> length result == 9)
  :+: Test "freqCount sum"
            (\x -> sum (map snd result) == 17)
```

- *What other tests make sense to specify?*
- *How are sum and snd defined?*

# Trees

We can represent a Huffmann tree as a user data type:

```
data Tree a = Leaf a
            | Tree a :^: Tree a
```

*Weigh a Tree*

```
weight :: Tree CharCount -> Int
weight (Leaf (ch,n))       = n
weight (tree1 :^: tree2)   = (weight tree1)
                             + (weight tree2)
```

# Testing Trees

Constructors are functions too:

```
map Leaf (freqCount iam)
  ➪ [ Leaf ('"',2), Leaf ('.',1), Leaf ('m',2),
    Leaf ('a',3), Leaf (' ',4), Leaf ('I',2),
    Leaf ('t',1), Leaf ('h',1), Leaf ('w',1) ]

map weight (map Leaf (freqCount iam))
                ➪ [ 2, 1, 2, 3, 4, 2, 1, 1, 1 ]

testWeight = Test "weight"
  (\x -> sum (map weight (map Leaf (freqCount iam)))
            == 17)
```

# Merging trees

*Recursively merge smallest trees together till a single tree results*

```
mergeTrees :: [Tree CharCount] -> Tree CharCount
mergeTrees [tree] = tree                -- base case
mergeTrees (tree1:tree2:treeList)  -- otherwise
  | w1 < w2          = mt treeList tree1 tree2 []
  | otherwise        = mt treeList tree2 tree1 []
      where {  w1 = (weight tree1);
               w2 = (weight tree2) }
```

We can decompose tree merging by means of a helper function ...

*Usage: mt untested tr1 tr2 tested, where weight(tr1) < weight(tr2) and tested is a list of trees with weights bigger than either tr1 or tr2*

```
mt [] tr1 tr2 []          = tr1 :^: tr2
mt [] tr1 tr2 tested  =
                        mergeTrees ((tr1 :^: tr2):tested)
mt (tr3:untested) tr1 tr2 tested
  | w3 < w1        = mt untested tr3 tr1 (tr2:tested)
  | w3 < w2        = mt untested tr1 tr3 (tr2:tested)
  | otherwise      = mt untested tr1 tr2 (tr3:tested)
      where {  w1 = (weight tr1); w2 = (weight tr2);
               w3 = (weight tr3) }
```

- *How do we know this terminates?*
- *Is there a more efficient way to merge trees?*

# Tree merging ...

```
mergeTrees (map Leaf (freqCount iam))
  ⇨ (  (  Leaf ('m',2)
           :^:
           ( Leaf ('w',1) :^: Leaf ('h',1) )
        )
        :^:
        (  ( Leaf ('.',1) :^: Leaf ('t',1) )
           :^:
           Leaf ('"',2)
        )
     )
     :^:
     (  Leaf (' ',4)
        :^:
        ( Leaf ('I',2) :^: Leaf ('a',3) )
     )
```

# Extracting the Huffmann tree

We remove the character counts to leave the Huffmann tree:

*Strip out the character counts from a Tree of CharCounts*

```
charTree :: Tree CharCount -> Tree Char
charTree (Leaf (ch,n))  = Leaf ch
charTree (tr1 :^: tr2)  = (charTree tr1)
                          :^: (charTree tr2)
```

*Generate an optimal Huffmann encoding tree for a given text*

```
huf :: String -> Tree Char
huf text = charTree (mergeTrees
                      (map Leaf (freqCount text)))
```

# Generating the tree

```
huf iam
      ➪ ( ( Leaf 'm'
              :^: ( Leaf 'w' :^: Leaf 'h'))
          :^: (( Leaf '.' :^: Leaf 't')
              :^: Leaf '"' ) )
        :^: ( Leaf ' '
          :^:
          ( Leaf 'I' :^: Leaf 'a'))
```

*NB: The resulting tree is not necessarily unique.*

# Extracting the encoding map

To encode text, we need to *store the path to each Char* in the tree:

```
mkEncode :: String -> (Tree Char) -> [(Char, String)]
mkEncode prefix (Leaf ch)      = [(ch, prefix)]
mkEncode prefix (tr1 :^: tr2) =
          (mkEncode (prefix ++ "0") tr1)
          ++ (mkEncode (prefix ++ "1") tr2)
```

```
mkEncode "" (huf iam)
  ➪ [('m',"000"), ('w',"0010"), ('h',"0011"),
    ('.',"0100"), ('t',"0101"), ('"',"011"),
    (' ',"10"), ('I',"110"), ('a',"111")]
```

# Applying the encoding map

To encode text, we just look up characters in the encoding map:

```
encChar :: [(Char, String)] -> Char -> String
encChar [] _    = undefined    -- shouldn't happen!
encChar ((ch,str):table) c
  | c == ch     = str
  | otherwise   = encChar table c

encode :: Tree Char -> String -> String
encode tree text    = foldr (++) ""
         (map (encChar (mkEncode "" tree)) text)
```

**encode (huf iam) iam** ➪

```
0111101011100010001000111110101101101011100001000011
```

# foldr

*NB:* `foldr` is defined in the standard prelude:

```
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr f z []= z
foldr f z (x:xs)= f x (foldr f z xs)
```

```
foldr (*) 1 [1..10]
```

➪ 3628800

# Decoding by walking the tree

To decode text, we just walk the tree, keeping a copy of the original tree so we can start over from the root each time we reach a leaf:

```
decode :: Tree Char -> String -> String
decode tree = walk tree tree     -- NB: higher order
walk :: Tree Char -> Tree Char -> String -> String
walk tree (tr1:^:tr2) ('0':rest) = walk tree tr1 rest
walk tree (tr1:^:tr2) ('1':rest) = walk tree tr2 rest
walk tree (Leaf ch) rest = [ch] ++ walk tree tree rest
walk tree nav [] = []

decode (huf iam) (encode (huf iam) iam)
➪ "\"I am what I am.\""
```

# Testing

*Test that decoding the encoded text yields the original:*

```
testHuf text = Test "huf encode/decode"
  (\x -> decode (huf text) (encode (huf text) text)
          == text)
```

```
assert (testHuf iam)
➪ PASSED all tests

assert (testHuf "")
➪ Program error: {mergeTrees []}
```

*Is this a reasonable thing to happen?*

# Representing trees as text

We need a way to *store Huffmann trees as plain text.*

We represent leaves by their character values, and intermediate nodes as *parenthesized expressions*, taking care to encode parentheses:

```
showTree :: Tree Char -> String
showTree (Leaf ch)
   | ch == '('     = "\\("
   | ch == ')'     = "\\)"
   | ch == '\\'    = "\\\\"
   | ch == '\n'    = "\\n"
   | otherwise     = [ch]
showTree (tr1 :^: tr2)= "(" ++ (showTree tr1)
                          ++ (showTree tr2) ++ ")"
...
```

# Representing trees as text ...

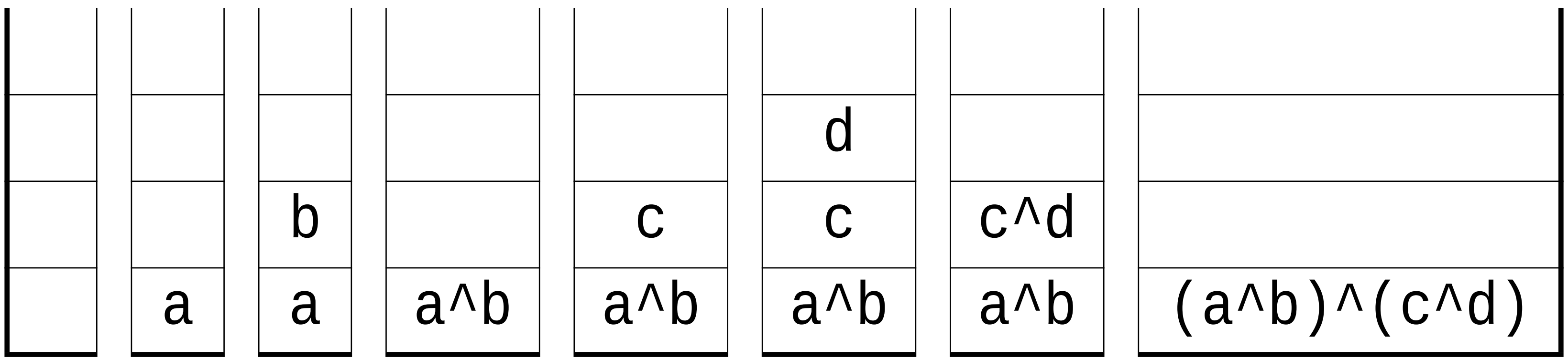
```
showTree (huf iam)
⇨ "(((m(wh))((.t)\"))( (Ia)))"

showTree (huf "()\\\n")
⇨ "((\\\\\\n)(\\(\\)))"

putStr (showTree (huf "()\\\n"))
⇨ ((\\\n)(\(\)))
```

# Using a stack to parse stored trees

Naturally, we need a way to *parse* and *reconstruct* the stored trees.

A standard solution is to push the leaves on a stack of trees, joining the top two elements every time a right parenthesis is encountered:

**Example:** `((ab)(cd))`




*If the parentheses are balanced, a single tree will be left on the stack.*

# Parsing stored trees

*Parse a Lisp-style parenthesized string, generating a Tree Char*

```
parseTree :: String -> Tree Char
parseTree       = pt [] -- initial stack is empty

pt :: [Tree Char] -> String -> Tree Char
pt [tree] []    = tree
pt stack (ch:str)
  | ch == '('   = pt stack str
  | ch == ')'   = pt (join stack) str
  | ch == '\\'  = pt
                  (Leaf (unescape (head str)):stack)
                  (tail str)
  | otherwise   = pt (Leaf ch:stack) str
```

# Parsing stored trees ...

*Join the top two trees of the stack into one*

```
join :: [Tree a] -> [Tree a]
join (tr1:tr2:stack)= (tr2:^:tr1):stack
```

*Unescape the character following a backslash*

```
unescape :: Char -> Char
unescape '('   = '('
unescape ')'   = ')'
unescape '\\'  = '\\'
unescape 'n'   = '\n'
```

```
parseTree (showTree (huf "()\\\n"))
⇨ (Leaf '\' :^: Leaf '\n') :^: (Leaf '(' :^: Leaf ')')
```

# Reading and Writing Files

Now we just need some functions to read the input file and write the result files:

*Reads a plain text file and generates the cipher and tree files*

```
enc :: FilePath -> IO ()
```

*Reads the cipher and tree files and regenerates the plain text*

```
dec :: FilePath -> IO()
```

There are standard libraries for dealing with user and file I/O.

✎ *How can you make sense of I/O in a purely functional world with no state changes?*

*See chapter 7 of "A Gentle Introduction to Haskell" for the complete story on IO!*

# Using the program (I)

From shell:

```
echo '"I am what I am."' > iam
```

From Haskell:

```
enc "iam"
```

From shell:

```
% cat iam.huf
➪ ((((\n.)(wh)) )((mI)((t")a)))
% cat iam.enc
➪11011010111110001001000111111100011010111110000011
1010000
```

✎ *Why do we get a different Hufmann encoding tree?*

# Using the program (II)

*Let's encode the source code of the program itself.*

From Haskell:

```
enc "huf"
⇨ (8598 reductions, 12940 cells)
INTERNAL ERROR: Application parameter stack overflow.
```

✎ *What went wrong?*

# Tracing our program

```
freqCount "abc"
⇨ incCount 'a' (freqCount "bc")
⇨ incCount 'a' (incCount 'b' (freqCount "c"))
⇨ incCount 'a' (incCount 'b' (incCount 'c' (freqCount "")))
⇨ incCount 'a' (incCount 'b' (incCount 'c' []))
⇨ incCount 'a' (incCount 'b' (('c',1) : []))
⇨ incCount 'a' (('c',1) : incCount 'b' [])
⇨ ('c',1) : incCount 'a' (incCount 'b' [])
⇨ ('c',1) : incCount 'a' (('b',1) : [])
⇨ ('c',1) : ('b',1) : incCount 'a' []
⇨ ('c',1) : ('b',1) : ('a',1) : []
```

Because Haskell is lazy, *nothing will happen until the entire input has been read*, thereby exhausting stack space for larger input files!

# Frequency Counting Revisited

We need frequency counting to be evaluated eagerly!

We can *force* evaluation by requiring values to be produced

*fcEager (c:s) front back -- front does not contain c, back to be checked*

```
fcEager :: String -> [CharCount] -> [CharCount]
                -> [CharCount]
fcEager "" [] ccl       = ccl

fcEager (c:s) front []= fcEager s [] ((c,1):front)

fcEager (c:s) front ((c1,n):back)
   | (c == c1) = fcEager s [] (front ++ ((c,n+1):back))
   | otherwise = fcEager (c:s) ((c1,n):front) back
```

# Tracing eager evaluation

```
fcEager "abc" [] []
⇨ fcEager “bc” [] (‘a’,1):[]                          -- new char
⇨ fcEager “bc” (‘a’,1):[] []                          -- ‘b’ != ‘a’
⇨ fcEager “c” [] (‘b’,1):(‘a’,1):[]                   -- new char
⇨ fcEager “c” (‘b’,1):[] (‘a’,1):[]                   -- ‘c’ != ‘b’
⇨ fcEager “c” (‘a’,1):(‘b’,1):[] []                   -- ‘c’ != ‘a’
⇨ fcEager “” [] (‘c’,1):(‘a’,1):(‘b’,1):[] []         -- base case
⇨ (‘c’,1):(‘a’,1):(‘b’,1):[] []                       -- ‘c’ != ‘a’
```

# Final version

```
fc2 s = fcEager s [] []      -- eager fc
enc2 = ...

enc2 "huf"
  ➪(2117457 reductions, 6145824 cells,
   100 garbage collections)
```

# What you should know!

- *How can you be sure a recursive function will terminate? How do we know that walk terminates?*
- *How do you know where characters end in Huffmann encoded bit strings?*
- *How can you generate a tree from its string representation?*
- *How can you force eager evaluation?*

# Can you answer these questions?

- *Can you prove that Huffmann's algorithm really generates the optimal map?*
- *What would happen if* encode *used foldl instead of foldr?*
- *Can parseTree be re-written so it uses the run-time stack instead of representing a stack as a list?*
- *Our Huffmann encoder actually outputs one byte for each "0" or "1"! How would you adapt the program to produce bits instead of bytes?*
- *Which functions implement the arrows in the architecture diagram?*

# 6. Introduction to the Lambda Calculus

**Overview**

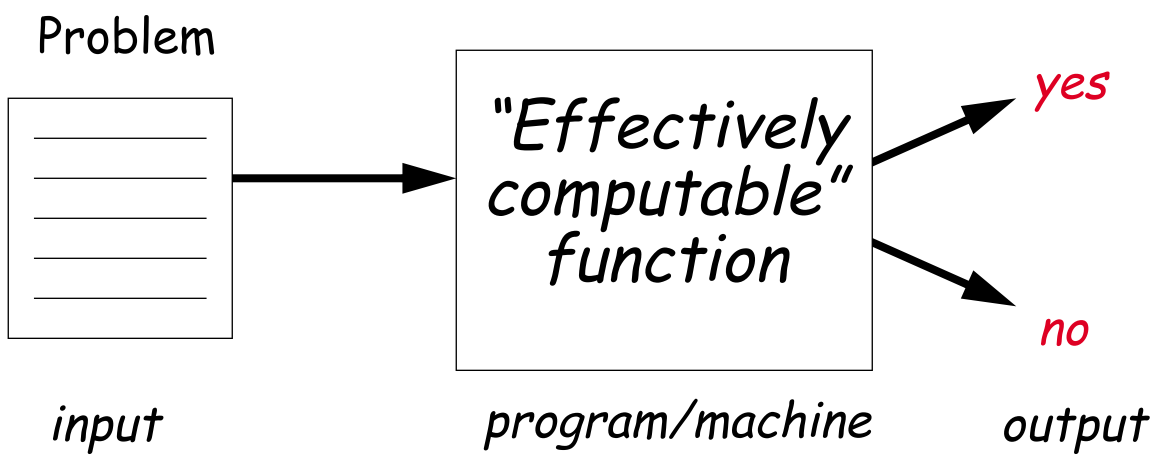
- What is Computability? — Church's Thesis
- Lambda Calculus — operational semantics
- The Church-Rosser Property
- Modelling basic programming constructs

# References


- Paul Hudak, "Conception, Evolution, and Application of Functional Programming Languages," ACM Computing Surveys 21/3, Sept. 1989, pp 359-411.
- Kenneth C. Louden, *Programming Languages: Principles and Practice*, PWS Publishing (Boston), 1993.
- H.P. Barendregt, *The Lambda Calculus — Its Syntax and Semantics*, North-Holland, 1984, Revised edition.

# What is Computable?

Computation is usually modelled as a *mapping* from *inputs* to *outputs*, carried out by a formal "*machine*," or program, which processes its input in a *sequence of steps*.




An "effectively computable" function is one that can be computed in a *finite amount of time* using *finite resources*.

# Church's Thesis

> *Effectively computable functions [from positive integers to positive integers] are just those definable in the lambda calculus.*

Or, equivalently:

> *It is not possible to build a machine that is more powerful than a Turing machine.*

Church's thesis cannot be proven because "effectively computable" is an *intuitive* notion, not a mathematical one. It can only be refuted by giving a counter-example — a machine that can solve a problem not computable by a Turing machine.

So far, *all* models of effectively computable functions have shown to be equivalent to Turing machines (or the lambda calculus).

# Uncomputability

A problem that cannot be solved by any Turing machine in finite time (or any equivalent formalism) is called *uncomputable*.

*Assuming Church's thesis is true, an uncomputable problem cannot be solved by any real computer.*

**The Halting Problem:**

> *Given an arbitrary Turing machine and its input tape, will the machine eventually halt?*

The Halting Problem is *provably uncomputable* — which means that it cannot be solved in practice.

# What is a Function? (I)

**Extensional view:**

A (total) *function* $f: A \rightarrow B$ is a *subset* of $A \times B$ (i.e., a *relation*) such that:

1. for each $a \in A$, there exists some $(a,b) \in f$
   (i.e., $f(a)$ is *defined*), and

2. if $(a,b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$
   (i.e., $f(a)$ is *unique*)

# What is a Function? (II)

**Intensional view:**

A <u>*function*</u> f: A $\rightarrow$ B is an *abstraction* $\lambda$ x . e, where x is a *variable name*, and e is an *expression*, such that when a value a$\in$ A is *substituted* for x in e, then this expression (i.e., f(a)) evaluates to some (unique) value b$\in$ B.

# The Lambda Calculus — syntax

The Lambda Calculus was invented by Alonzo Church [1932] as a mathematical formalism for expressing computation by functions.

**Syntax:**

| | | | |
|---|---|---|---|
| $e$ | $::=$ | $x$ | *a variable* |
| | $\mid$ | $\lambda\, x\, .\, e$ | *an abstraction (function)* |
| | $\mid$ | $e_1\, e_2$ | *a (function) application* |

$\lambda\, x\, .\, x$ — is a function taking an argument $x$, and returning $x$

# Lambda Calculus — semantics

**(Operational) Semantics:**

| | | |
|---|---|---|
| *$\alpha$ conversion (renaming):* | $\lambda\, x\, .\, e \leftrightarrow \lambda\, y\, .\, [\, y/x\, ]\, e$ | *where y is not free in e* |
| *$\beta$ reduction (application):* | $(\lambda\, x\, .\, e_1)\, e_2 \rightarrow [\, e_2/x\, ]\, e_1$ | *avoiding name capture* |
| *$\eta$ reduction:* | $\lambda\, x\, .\, (e\, x) \rightarrow e$ | *if x is not free in e* |

The lambda calculus can be viewed as the simplest possible pure functional programming language.

# Beta Reduction

Beta reduction is the computational engine of the lambda calculus:

Define: $I \equiv \lambda x . x$

Now consider:

$$
\begin{aligned}
I\,I = (\lambda x . x)(\lambda x . x) \quad & \rightarrow \quad [(\lambda x . x) / x]\, x && \textit{\beta reduction} \\
& = \quad (\lambda x . x) && \textit{substitution} \\
& = \quad I
\end{aligned}
$$

# Lambda expressions in Haskell

We can implement most lambda expressions directly in Haskell:

```
i = \x -> x
? i 5
5
(2 reductions, 6 cells)
? i i 5
5
(3 reductions, 7 cells)
```

# Free and Bound Variables

The variable x is *bound* by $\lambda$ in the expression: $\lambda$ x.e
A variable that is not bound, is *free* :

$$\begin{aligned} fv(x) &= \{ x \} \\ fv(e_1\ e_2) &= fv(e_1) \cup fv(e_2) \\ fv(\lambda\ x\ .\ e) &= fv(e) - \{ x \} \end{aligned}$$

An expression with *no free variables* is *closed*.
(AKA a *combinator*.) Otherwise it is *open*.

For example, y is *bound* and x is *free* in the (open) expression:
$\lambda$ y . x y

# Why macro expansion is wrong

*Syntactic substitution will not work:*

$$(\lambda x . \lambda y . x\, y)\, y \rightarrow [y / x]\, (\lambda y . x\, y)$$ *β reduction*

$$\neq (\lambda y . y\, y)$$ *incorrect substitution!*

Since y is *already bound* in $(\lambda y . x\, y)$, we *cannot* directly substitute y for x.

# Substitution

We must define substitution carefully to avoid *name capture:*

$$[e/x]\ x = e$$

$$[e/x]\ y = y \qquad \textit{if } x \neq y$$

$$[e/x]\ (e_1\ e_2) = ([e/x]\ e_1)\ ([e/x]\ e_2)$$

$$[e/x]\ (\lambda\ x\ .\ e_1) = (\lambda\ x\ .\ e_1)$$

$$[e/x]\ (\lambda\ y\ .\ e_1) = (\lambda\ y\ .\ [e/x]\ e_1) \qquad \textit{if } x \neq y \textit{ and } y \notin fv(e)$$

$$[e/x]\ (\lambda\ y\ .\ e_1) = (\lambda\ z\ .\ [e/x]\ [z/y]\ e_1) \qquad \textit{if } x \neq y \textit{ and } z \notin fv(e) \cup fv(e_1)$$

*Consider:*

$$(\lambda\ x\ .\ ((\lambda\ y\ .\ x)\ (\lambda\ x\ .\ x))\ x\ )\ y \rightarrow [y\ /\ x]\ ((\lambda\ y\ .\ x\ )\ (\lambda\ x\ .\ x))\ x$$

$$=\ ((\lambda\ z\ .\ y\ )\ (\lambda\ x\ .\ x))\ y$$

# Alpha Conversion

Alpha conversions allows us to *rename bound variables.*

A bound name x in the lambda abstraction ($\lambda$ x.e) may be substituted by any other name y, as long as there are *no free occurrences of y in e:*

Consider:

$$
\begin{array}{llll}
(\lambda x . \lambda y . x\, y)\, y & \rightarrow & (\lambda x . \lambda z . x\, z)\, y & \alpha \text{ conversion} \\
 & \rightarrow & [y / x]\, (\lambda z . x\, z) & \beta \text{ reduction} \\
 & \rightarrow & (\lambda z . y\, z) & \\
 & = & y & \eta \text{ reduction}
\end{array}
$$

# Eta Reduction

Eta reductions allows one to remove "redundant lambdas".

Suppose that f is a *closed expression*
(i.e., there are no free variables in f).

Then:

$$( \lambda x . f x ) y \quad \rightarrow f y \qquad \textit{\beta reduction}$$

So, ( $\lambda$ x . f x ) behaves the same as f !

Eta reduction says, *whenever x does not occur free in f*, we can rewrite ( $\lambda$ x . f x ) as f.

# Normal Forms

A lambda expression is in <u>*normal form*</u> if it can no longer be reduced by beta or eta reduction rules.

*Not all lambda expressions have normal forms!*

$$
\begin{aligned}
\Omega = (\lambda x . x\ x)(\lambda x . x\ x) &\to [\,(\lambda x . x\ x) / x\,]\,(x\ x) \\
&= (\lambda x . x\ x)(\lambda x . x\ x) && \textit{\beta reduction} \\
&\to (\lambda x . x\ x)(\lambda x . x\ x) && \textit{\beta reduction} \\
&\to (\lambda x . x\ x)(\lambda x . x\ x) && \textit{\beta reduction} \\
&\to \ldots
\end{aligned}
$$

Reduction of a lambda expression to a normal form is analogous to a *Turing machine halting* or a *program terminating*.

# Evaluation Order

Most programming languages are *strict*, that is, all expressions passed to a function call are *evaluated before control is passed to the function.*

Most modern functional languages, on the other hand, use *lazy* evaluation, that is, expressions are *only evaluated when they are needed.*

*Consider:*

```
sqr n = n * n
```

Applicative-order reduction:

```
sqr (2+5) ⇨ sqr 7 ⇨ 7*7 ⇨ 49
```

Normal-order reduction:

```
sqr (2+5) ⇨ (2+5) * (2+5) ⇨ 7 * (2+5) ⇨ 7 * 7 ⇨ 49
```

# The Church-Rosser Property

*"If an expression can be evaluated at all, it can be evaluated by consistently using normal-order evaluation. If an expression can be evaluated in several different orders (mixing normal-order and applicative order reduction), then all of these evaluation orders yield the same result."*

So, evaluation order "does not matter" in the lambda calculus.

# Non-termination

*However, applicative order reduction may not terminate, even if a normal form exists!*

$$(\lambda x\,.\,y)\,((\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x))$$

| *Applicative order reduction* | *Normal order reduction* |
| --- | --- |
| $\rightarrow (\lambda x\,.\,y)\,((\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x))$ | $\rightarrow y$ |
| $\rightarrow (\lambda x\,.\,y)\,((\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x))$ | |
| $\rightarrow \ldots$ | |

*Compare to the Haskell expression:*

**(\x -> \y -> x) 1 (5/0)** ⇨ 1

# Currying

Since a lambda abstraction only binds a single variable, functions with multiple parameters must be modelled as *Curried* higher-order functions.

To improve readability, *multiple lambdas can be suppressed*, so:

$$\lambda\, x\, y\, .\, x = \lambda\, x\, .\, \lambda\, y\, .\, x$$
$$\lambda\, b\, x\, y\, .\, b\, x\, y = \lambda\, b\, .\, \lambda\, x\, .\, \lambda\, y\, .\, (\, b\, x\, )\, y$$

# Representing Booleans

Many programming concepts can be directly expressed in the lambda calculus. *Let us define:*

$$\begin{aligned}
\text{True} &\equiv \lambda\, x\, y\, .\, x \\
\text{False} &\equiv \lambda\, x\, y\, .\, y \\
\text{not} &\equiv \lambda\, b\, .\, b\ \text{False}\ \text{True} \\
\text{if } b \text{ then } x \text{ else } y &\equiv \lambda\, b\, x\, y\, .\, b\, x\, y
\end{aligned}$$

*then:*

$$\begin{aligned}
\text{not True} &= (\, \lambda\, b\, .\, b\ \text{False True}\,)\, (\, \lambda\, x\, y\, .\, x\,) \\
&\to (\, \lambda\, x\, y\, .\, x\,)\ \text{False True} \\
&\to \text{False} \\
\text{if True then } x \text{ else } y &= (\, \lambda\, b\, x\, y\, .\, b\, x\, y\,)\, (\lambda\, x\, y\, .\, x)\, x\, y \\
&\to (\lambda\, x\, y\, .\, x)\, x\, y \\
&\to x
\end{aligned}$$

# Representing Tuples

Although tuples are not supported by the lambda calculus, they can easily be modelled as *higher-order functions* that "*wrap*" pairs of values.

n-tuples can be modelled by composing pairs ...

*Define:*

$$
\begin{aligned}
\text{pair} &\equiv (\lambda\, x\, y\, z\, .\, z\, x\, y) \\
\text{first} &\equiv (\lambda\, p\, .\, p\ \text{True}\ ) \\
\text{second} &\equiv (\lambda\, p\, .\, p\ \text{False}\ )
\end{aligned}
$$

*then:*

$$
\begin{aligned}
(1, 2) &= \text{pair}\ 1\ 2 \\
&\rightarrow (\lambda\, z\, .\, z\ 1\ 2) \\
\text{first}\ (\text{pair}\ 1\ 2) &\rightarrow (\text{pair}\ 1\ 2)\ \text{True} \\
&\rightarrow \text{True}\ 1\ 2 \\
&\rightarrow 1
\end{aligned}
$$

# Tuples as functions

In Haskell:

```
t      = \x -> \y -> x
f      = \x -> \y -> y
pair   = \x -> \y -> \z -> z x y
first  = \p -> p t
second = \p -> p f
? first (pair 1 2)
1
? first (second (pair 1 (pair 2 3)))
2
```

# Representing Numbers

There is a "standard encoding" of natural numbers into the lambda calculus:

*Define:*

$$0 \equiv (\lambda\, x \,.\, x\,)$$
$$\text{succ} \equiv (\lambda\, n \,.\, (\text{False}, n)\,)$$

*then:*

$$1 \equiv \text{succ } 0 \quad \rightarrow \quad (\text{False}, 0)$$
$$2 \equiv \text{succ } 1 \quad \rightarrow \quad (\text{False}, 1)$$
$$3 \equiv \text{succ } 2 \quad \rightarrow \quad (\text{False}, 2)$$
$$4 \equiv \text{succ } 3 \quad \rightarrow \quad (\text{False}, 3)$$

...

# Working with numbers

We can define simple functions to work with our numbers.

*Consider:*

$$\text{iszero} \equiv \text{first}$$
$$\text{pred} \equiv \text{second}$$

*then:*

$$\text{iszero } 1 = \text{first (False, 0)} \quad \rightarrow \text{False}$$
$$\text{iszero } 0 = (\lambda\, p\, .\, p \text{ True })(\lambda\, x\, .\, x\,) \quad \rightarrow \text{True}$$
$$\text{pred } 1 = \text{second (False, 0)} \quad \rightarrow 0$$

✎ *What happens when we apply pred 0? What does this mean?*

# What you should know!

- *Is it possible to write a Pascal compiler that will generate code just for programs that terminate?*
- *What are the alpha, beta and eta conversion rules?*
- *What is name capture? How does the lambda calculus avoid it?*
- *What is a normal form? How does one reach it?*
- *What are normal and applicative order evaluation?*
- *Why is normal order evaluation called lazy?*
- *How can Booleans, tuples and numbers be represented in the lambda calculus?*

# Can you answer these questions?

- *How can name capture occur in a programming language?*
- *What happens if you try to program Ω in Haskell? Why?*
- *What do you get when you try to evaluate (pred 0)? What does this mean?*
- *How would you model negative integers in the lambda calculus? Fractions?*
- *Is it possible to model real numbers? Why, or why not?*

# 7. Fixed Points and other Calculi

**Overview**

- ❑ Recursion and the Fixed-Point Combinator
- ❑ The typed lambda calculus
- ❑ The polymorphic lambda calculus
- ❑ A quick look at process calculi

**References:**

- ❑ Paul Hudak, "Conception, Evolution, and Application of Functional Programming Languages," ACM Computing Surveys 21/3, Sept. 1989, pp 359-411.

# Recursion

Suppose we want to define *arithmetic operations* on our lambda-encoded numbers.

In Haskell we can program:

```
plus n m
  | n == 0      = m
  | otherwise   = plus (n-1) (m+1)
```

so we might try to "define":

$$\text{plus} \equiv \lambda\, n\ m\ .\ \text{iszero}\ n\ m\ (\ \text{plus}\ (\ \text{pred}\ n\ )\ (\ \text{succ}\ m\ )\ )$$

Unfortunately this is *not a definition*, since we are trying to *use plus before it is defined.* I.e, plus is free in the "definition"!

# Recursive functions as fixed points

We can obtain a *closed expression* by *abstracting* over plus:

```
rplus ≡ λ plus n m . iszero n
                m
                ( plus ( pred n ) ( succ m ) )
```

rplus takes as its *argument* the actual plus function to use and returns as its result a definition of that function in terms of itself. In other words, if fplus is the function we want, then:

$$\text{rplus fplus} \leftrightarrow \text{fplus}$$

I.e., we are searching for a *fixed point* of rplus ...

# Fixed Points

A *fixed point* of a function `f` is a value `p` such that `f p = p`.

**Examples:**

```
fact 1 = 1
fact 2 = 2
fib 0  = 0
fib 1  = 1
```

Fixed points are not always "well-behaved":

```
succ n = n + 1
```

✎ *What is a fixed point of* `succ`*?*

# Fixed Point Theorem

**Theorem:**
Every lambda expression e has a *fixed point* p such that $(e\ p) \leftrightarrow p$.

**Proof:** Let:

$$Y \equiv \lambda f\,.\,(\lambda x\,.\,f\,(x\ x))\,(\lambda x\,.\,f\,(x\ x))$$

Now consider:

$$\begin{aligned} p \equiv Y\ e &\rightarrow (\lambda\ x\,.\,e\,(x\ x))\,(\lambda x\,.\,e\,(x\ x)) \\ &\rightarrow e\,((\lambda x\,.\,e\,(x\ x))\,(\lambda x\,.\,e\,(x\ x))) \\ &= e\ p \end{aligned}$$

So, the "magical Y combinator" can always be used to find a fixed point of an *arbitrary* lambda expression.

# Using the Y Combinator

Consider:

$$f \equiv \lambda\, x.\ \mathrm{True}$$

then:

$$\begin{aligned} Y\ f &\rightarrow f\ (Y\ f) && \textit{by FP theorem} \\ &= (\lambda\, x.\ \mathrm{True})\ (Y\ f) \\ &\rightarrow \mathrm{True} \end{aligned}$$

Consider:

$$\begin{aligned} Y\ succ &\rightarrow succ\ (Y\ succ) && \textit{by FP theorem} \\ &\rightarrow (\mathrm{False},\ (Y\ succ)) \end{aligned}$$

✎ *What are succ and pred of (False, (Y succ))? What does this represent?*

# Recursive Functions are Fixed Points

*We seek a fixed point of:*

$$\text{rplus} \equiv \lambda \text{ plus n m . iszero n m ( plus ( pred n ) ( succ m ) )}$$

By the Fixed Point Theorem, we simply take:

$$\text{plus} \equiv \text{Y rplus}$$

Since this guarantees that:

$$\text{rplus plus} \leftrightarrow \text{plus}$$

*as desired!*

# Unfolding Recursive Lambda Expressions

plus 1 1 = (Y rplus) 1 1
→ rplus plus 1 1
→ iszero 1 1 (plus (pred 1) (succ 1) )
→ False 1 (plus (pred 1) (succ 1) )
→ plus (pred 1) (succ 1)
→ rplus plus (pred 1) (succ 1)
→ iszero (pred 1) (succ 1)
    (plus (pred (pred 1) ) (succ (succ 1) ) )
→ iszero 0 (succ 1) (...)
→ True (succ 1) (...)
→ succ 1
→ 2

# The Typed Lambda Calculus

There are many variants of the lambda calculus.
The *typed lambda calculus* just decorates terms with *type annotations:*

**Syntax:** $e ::= x^{\tau} \mid e_1^{\tau 2 \rightarrow \tau 1}\ e_2^{\tau 2} \mid (\lambda\, x^{\tau 2}.e^{\tau 1})^{\tau 2 \rightarrow \tau 1}$

**Operational Semantics:**

$$\lambda\, x^{\tau 2} \,.\, e^{\tau 1} \Leftrightarrow \lambda\, y^{\tau 2} \,.\, [\, y^{\tau 2}/x^{\tau 2}\,]\, e^{\tau 1} \qquad y^{\tau 2} \text{ not free in } e^{\tau 1}$$

$$(\lambda\, x^{\tau 2} \,.\, e_1^{\tau 1})\, e_2^{\tau 2} \Rightarrow [\, e_2^{\tau 2}/x^{\tau 2}\,]\, e_1^{\tau 1}$$

$$\lambda\, x^{\tau 2}.\, (e^{\tau 1}\, x^{\tau 2}) \Rightarrow e^{\tau 1} \qquad x^{\tau 2} \text{ not free in } e^{\tau 1}$$

*Example:*

$$\text{True} \equiv (\,\lambda\, x^{A} \,.\, (\,\lambda\, y^{B} \,.\, x^{A}\,)^{B \rightarrow A})^{A \rightarrow (B \rightarrow A)}$$

# The Polymorphic Lambda Calculus

Polymorphic functions like "map" cannot be typed in the typed lambda calculus!

Need *type variables* to capture polymorphism:

$\beta$ reduction (ii): $(\lambda\, x^{\nu}\, .\, e_1^{\tau 1})\, e_2^{\tau 2} \Rightarrow [\, \tau 2 \,/\, \nu \,]\, [\, e_2^{\tau 2} / x^{\nu}\, ]\, e_1^{\tau 1}$

*Example:*

$$
\begin{aligned}
\mathrm{True} &\equiv (\, \lambda\, x^{\alpha}\, .\, (\, \lambda\, y^{\beta}\, .\, x^{\alpha}\, )^{\beta\rightarrow\alpha})^{\alpha\rightarrow(\beta\rightarrow\alpha)} \\
\mathrm{True}^{\alpha\rightarrow(\beta\rightarrow\alpha)}\, a^{A}\, b^{B} &\rightarrow (\, \lambda\, y^{\beta}\, .\, a^{A}\, )^{\beta\rightarrow A}\, b^{B} \\
&\rightarrow\ a^{A}
\end{aligned}
$$

# Hindley-Milner Polymorphism

Hindley-Milner polymorphism (i.e., that adopted by ML and Haskell) works by inferring the type annotations for a slightly restricted subcalculus: polymorphic functions.

If:

```
doubleLen len len' xs ys = (len xs) + (len' ys)
```

then

```
doubleLen length length “aaa” [1,2,3]
```

is ok, but if

```
doubleLen' len xs ys = (len xs) + (len ys)
```

then

```
doubleLen' length “aaa” [1,2,3]
```

is a type error since the argument `len` cannot be assigned a *unique* type!

# Polymorphism and self application

*Even the polymorphic lambda calculus is not powerful enough to express certain lambda terms.*

Recall that both $\Omega$ and the Y combinator make use of "self application":

$$\Omega = (\lambda\, x\, .\, x\, x\, )\, (\lambda\, x\, .\, x\, x\, )$$

✎ *What type annotation would you assign to* $(\lambda\, x\, .\, x\, x)$*?*

# Other Calculi

Many calculi have been developed to study the semantics of programming languages.

**Object calculi:** model *inheritance* and *subtyping* ..

☞ lambda calculi with records

**Process calculi:** model *concurrency* and *communication*

☞ CSP, CCS, π calculus, CHAM, blue calculus

**Distributed calculi:** model *location* and *failure*

☞ ambients, join calculus

# What you should know!

- *Why isn't it possible to express recursion directly in the lambda calculus?*
- *What is a fixed point? Why is it important?*
- *How does the typed lambda calculus keep track of the types of terms?*
- *How does a polymorphic function differ from an ordinary one?*

# Can you answer these questions?

- *Are there more fixed-point operators other than Y?*
- *How can you be sure that unfolding a recursive expression will terminate?*
- *Would a process calculus be Church-Rosser?*

# 8. Introduction to Denotational Semantics

**Overview:**

- Syntax and Semantics
- Approaches to Specifying Semantics
- Semantics of Expressions
- Semantics of Assignment
- Other Issues

**References:**


- D. A. Schmidt, *Denotational Semantics*, Wm. C. Brown Publ., 1986
- D. Watt, *Programming Language Concepts and Paradigms*, Prentice Hall, 1990

# Defining Programming Languages

**Three main characteristics of programming languages:**

1. **Syntax:** What is the *appearance* and *structure* of its programs?
2. **Semantics:** What is the *meaning* of programs?

   The *static semantics* tells us which (syntactically valid) programs are semantically valid (i.e., which are *type correct*) and the *dynamic semantics* tells us how to interpret the meaning of valid programs.
3. **Pragmatics:** What is the *usability* of the language?

   How *easy is it to implement*? What kinds of applications does it suit?

# Uses of Semantic Specifications

Semantic specifications are useful for language designers to communicate with implementors as well as with programmers.

**A precise standard for a computer implementation:**

How should the language be *implemented* on different machines?

**User documentation:** What is the *meaning* of a program, given a particular combination of language features?

**A tool for design and analysis:** How can the language definition be *tuned* so that it can be implemented *efficiently*?

**Input to a compiler generator:** How can a *reference implementation* be obtained from the specification?

# Methods for Specifying Semantics

**Operational Semantics:**

- ☞ ⟦ program ⟧ = *abstract machine program*
- ☞ can be simple to implement
- ☞ hard to reason about

**Denotational Semantics:**

- ☞ ⟦ program ⟧ = *mathematical denotation* (typically, a function)
- ☞ facilitates reasoning
- ☞ not always easy to find suitable semantic domains

...

# Methods for Specifying Semantics ...

**Axiomatic Semantics:**

- ⟦ program ⟧ = *set of properties*
- good for proving theorems about programs
- somewhat distant from implementation

**Structured Operational Semantics:**

- ⟦ program ⟧ = *transition system*
  (defined using inference rules)
- good for concurrency and non-determinism
- hard to reason about equivalence

# Concrete and Abstract Syntax

How to parse "4 * 2 + 1"?

*Abstract Syntax* is compact but ambiguous:

```
Expr    ::= Num | Expr Op Expr
Op      ::= + | - | * | /
```

*Concrete Syntax* is unambiguous but verbose:

```
Expr    ::= Expr LowOp Term | Term
Term    ::= Term HighOp Factor | Factor
Factor  ::= Num | ( Expr )
LowOp   ::= + | -
HighOp  ::= * | /
```

*Concrete syntax is needed for parsing; abstract syntax suffices for semantic specifications.*

# A Calculator Language

**Abstract Syntax:**

```
Prog  ::=  'ON' Stmt
Stmt  ::=  Expr 'TOTAL' Stmt
        |  Expr 'TOTAL' 'OFF'
Expr  ::=  Expr1 '+' Expr2
        |  Expr1 '*' Expr2
        |  'IF' Expr1 ',' Expr2 ',' Expr3
        |  'LASTANSWER'
        |  '(' Expr ')'
        |  Num
```

The program "`ON 4 * ( 3 + 2 ) TOTAL OFF`" should print out `20` and stop.

# Calculator Semantics

We need three semantic functions: one for *programs*, one for *statements* (expression sequences) and one for *expressions*.

*The meaning of a program is the list of integers printed:*

**Programs:**

$$\mathbf{P} : \mathit{Program} \rightarrow \mathit{Int}^{\,*}$$

$$\mathbf{P}\,[\![\ \mathrm{ON}\ \mathrm{S}\ ]\!] \;=\; \mathbf{S}\,[\![\ \mathrm{S}\ ]\!]\,(0)$$

*A statement may use and update LASTANSWER:*

**Statements:**

$$\mathbf{S} :: \mathit{ExprSequence} \rightarrow \mathit{Int} \rightarrow \mathit{Int}^{\,*}$$

$$\mathbf{S}\,[\![\ \mathrm{E}\ \mathrm{TOTAL}\ \mathrm{S}\ ]\!]\,(n) \;=\; \mathit{let}\ n' = \mathbf{E}\,[\![\ \mathrm{E}\ ]\!]\,(n)\ \ \mathit{in}\ \mathrm{cons}(n',\,\mathbf{S}\,[\![\ \mathrm{S}\ ]\!]\,(n'))$$

$$\mathbf{S}\,[\![\ \mathrm{E}\ \mathrm{TOTAL}\ \mathrm{OFF}\ ]\!]\,(n) \;=\; [\ \mathbf{E}\,[\![\ \mathrm{E}\ ]\!]\,(n)\ ]$$

# Calculator Semantics...

## Expressions:

$$\mathbf{E} : \mathit{Expression} \rightarrow \mathit{Int} \rightarrow \mathit{Int}$$

$$\begin{aligned}
\mathbf{E} [\![ \text{E1} + \text{E2} ]\!] (\text{n}) &= \mathbf{E} [\![ \text{E1} ]\!] (\text{n}) + \mathbf{E} [\![ \text{E2} ]\!] (\text{n}) \\
\mathbf{E} [\![ \text{E1} * \text{E2} ]\!] (\text{n}) &= \mathbf{E} [\![ \text{E1} ]\!] (\text{n}) \times \mathbf{E} [\![ \text{E2} ]\!] (\text{n}) \\
\mathbf{E} [\![ \texttt{IF}\ \text{E1} , \text{E2} , \text{E3} ]\!] (\text{n}) &= \mathit{if}\ \mathbf{E} [\![ \text{E1} ]\!] (\text{n}) = 0 \\
&\quad\ \mathit{then}\ \mathbf{E} [\![ \text{E2} ]\!] (\text{n}) \\
&\quad\ \mathit{else}\ \mathbf{E} [\![ \text{E3} ]\!] (\text{n}) \\
\mathbf{E} [\![ \text{LASTANSWER} ]\!] (\text{n}) &= \text{n} \\
\mathbf{E} [\![ ( \text{E} ) ]\!] (\text{n}) &= \mathbf{E} [\![ \text{E} ]\!] (\text{n}) \\
\mathbf{E} [\![ \text{N} ]\!] (\text{n}) &= \text{N}
\end{aligned}$$

# Semantic Domains

In order to define semantic mappings of programs and their features to their mathematical denotations, the semantic domains must be precisely defined:

```
data Bool = True | False
(&&), (||) :: Bool -> Bool -> Bool
False   &&  x   = False
True    &&  x   = x
False   ||  x   = x
True    ||  x   = True

not :: Bool -> Bool
not     True    = False
not     False   = True
```

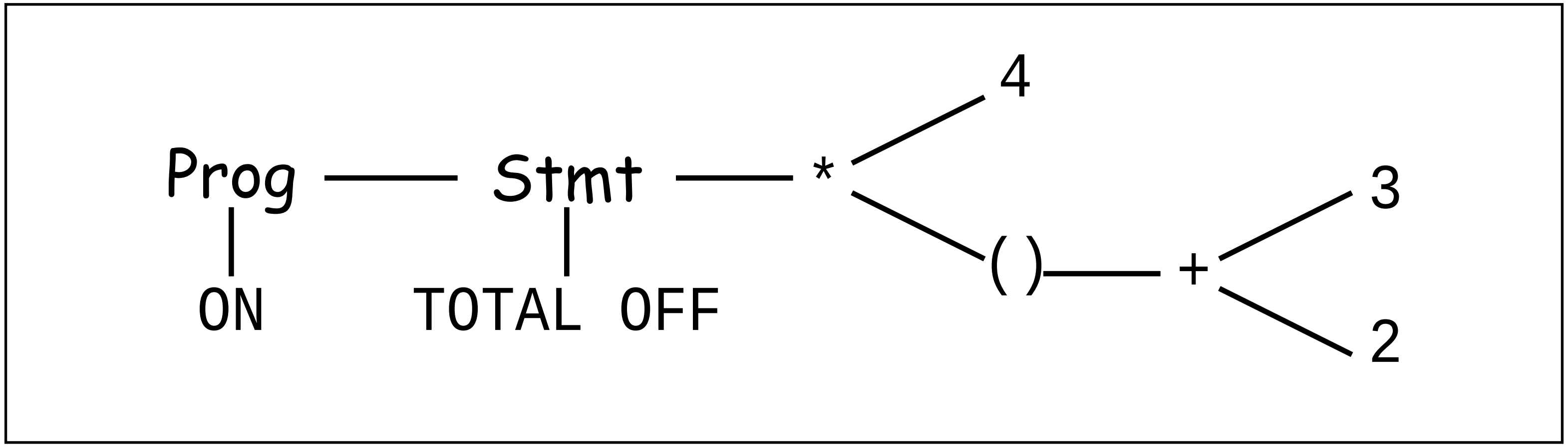
# Data Structures for Abstract Syntax

We can represent programs in our calculator language as syntax trees:

```
data Program = On ExprSequence
data ExprSequence = Total Expression ExprSequence
            | TotalOff Expression
data Expression = Plus Expression Expression
            | Times Expression Expression
            | If Expression Expression Expression
            | LastAnswer
            | Braced Expression
            | N Int
```

# Representing Syntax

The test program "`ON 4 * ( 3 + 2 ) TOTAL OFF`" can be *parsed* as:




And *represented* as:

```
test = On (TotalOff (Times (N 4)
                          (Braced (Plus (N 3)
                                        (N 2))))))
```

# Implementing the Calculator

*We can implement our denotational semantics directly in a functional language like Haskell:*

**Programs:**

```
pp :: Program -> [Int]
pp (On s)             = ss s 0
```

**Statements:**

```
ss :: ExprSequence -> Int -> [Int]
ss (Total e s) n   = let n' = (ee e n)
                       in n' : (ss s n')
ss (TotalOff e) n = (ee e n) : [ ]
```

...

# Implementing the Calculator ...

**Expressions:**

```
ee :: Expression -> Int -> Int
ee (Plus e1 e2) n    = (ee e1 n) + (ee e2 n)
ee (Times e1 e2) n   = (ee e1 n) * (ee e2 n)
ee (If e1 e2 e3) n
  | (ee e1 n) == 0   = (ee e2 n)
  | otherwise        = (ee e3 n)
ee (LastAnswer) n    = n
ee (Braced e) n      = (ee e n)
ee (N num) n         = num
```

# A Language with Assignment

| | | |
|---|---|---|
| Prog | ::= | Cmd '.' |
| Cmd | ::= | $Cmd_1$ ';' $Cmd_2$ |
| | \| | 'if' Bool 'then' $Cmd_1$ 'else' $Cmd_2$ |
| | \| | Id ':=' Exp |
| Exp | ::= | $Exp_1$ '+' $Exp_2$ |
| | \| | Id |
| | \| | Num |
| Bool | ::= | $Exp_1$ '=' $Exp_2$ |
| | \| | 'not' Bool |

**Example:**

“z := 1 ; if a = 0 then z := 3 else z := z + a .”

*Input number initializes a; output is final value of z.*

# Representing abstract syntax trees

**Data Structures:**

```
data Program        =   Dot Command
data Command        =   CSeq Command Command
                    |   Assign Identifier Expression
                    |   If BooleanExpr Command Command
data Expression     =   Plus Expression Expression
                    |   Id Identifier
                    |   Num Int
data BooleanExpr    =   Equal Expression Expression
                    |   Not BooleanExpr
type Identifier     =   Char
```

# An abstract syntax tree

**Example:**

“z := 1 ; if a = 0 then z := 3 else z := z + a .”

*Is represented as:*

```
Dot   (CSeq (Assign 'z' (Num 1))
            (If (Equal (Id 'a') (Num 0))
                (Assign 'z' (Num 3))
                (Assign 'z' (Plus (Id 'z') (Id 'a')))
            )
      )
```

# Modelling Environments

*A store is a mapping from identifiers to values:*

```
type Store = Identifier -> Int
newstore :: Store
newstore id         =   0

update :: Identifier -> Int -> Store -> Store
update id val store    =   store'
                            where store' id'
                            | id' == id = val
                            | otherwise = store id'
```

# Functional updates

*Example:*

```
env1 = update 'a' 1 (update 'b' 2 (newstore))
env2 = update 'b' 3 env1
```

```
env1 ‘b’
```
➪ 2

```
env2 ‘b’
```
➪ 3

```
env2 ‘z’
```
➪ 0

# Semantics of assignments

```
pp :: Program -> Int -> Int
pp (Dot c) n = (cc c (update 'a' n newstore)) ‘z’

cc :: Command -> Store -> Store
cc (CSeq c1 c2) s    = cc c2 (cc c1 s)
cc (Assign id e) s   = update id (ee e s) s
cc (If b c1 c2) s    = ifelse (bb b s)
                         (cc c1 s) (cc c2 s)
```

…

# Semantics of assignments ...

```
ee :: Expression -> Store -> Int
ee (Plus e1 e2) s  = (ee e2 s) + (ee e1 s)
ee (Id id) s       = s id
ee (Num n) s       = n

bb :: BooleanExpr -> Store -> Bool
bb (Equal e1 e2) s = (ee e1 s) == (ee e2 s)
bb (Not b) s       = not (bb b s)

ifelse :: Bool -> a -> a -> a
ifelse True x y    = x
ifelse False x y   = y
```

# Running the interpreter

```
src1 = "z := 1 ; if a = 0 then z := 3 else z := z + a ."
ast1 = Dot (CSeq
        (Assign 'z' (Num 1))
          (If (Equal (Id 'a') (Num 0))
            (Assign 'z' (Num 3))
            (Assign 'z' (Plus (Id 'z') (Id 'a')))))
```

**pp ast1 10**

⇨ 11

# Practical Issues

**Modelling:**

- Errors and non-termination:
  - ☞ need a special "error" value in semantic domains
- Branching:
  - ☞ semantic domains in which "continuations" model "the rest of the program" make it easy to transfer control
- Interactive input
- Dynamic typing
- ...

# Theoretical Issues

*What are the denotations of lambda abstractions?*

- need Scott's theory of semantic domains

*What is the semantics of recursive functions?*

- need least fixed point theory

*How to model concurrency and non-determinism?*

- abandon standard semantic domains
- use "interleaving semantics"
- "true concurrency" requires other models ...

# What you should know!

- *What is the difference between syntax and semantics?*
- *What is the difference between abstract and concrete syntax?*
- *What is a semantic domain?*
- *How can you specify semantics as mappings from syntax to behaviour?*
- *How can assignments and updates be modelled with (pure) functions?*

# Can you answer these questions?

- *Why are semantic functions typically higher-order?*
- *Does the calculator semantics specify strict or lazy evaluation?*
- *Does the implementation of the calculator semantics use strict or lazy evaluation?*
- *Why do commands and expressions have different semantic domains?*

# 9. Logic Programming

**Overview**

- Facts and Rules
- Resolution and Unification
- Searching and Backtracking
- Recursion, Functions and Arithmetic
- Lists and other Structures

# References


- Kenneth C. Louden, *Programming Languages: Principles and Practice*, PWS Publishing (Boston), 1993.
- Sterling and Shapiro, *The Art of Prolog*, MIT Press, 1986
- Clocksin and Mellish, *Programming in Prolog*, Springer Verlag, 1981

# Logic Programming Languages

**What is a Program?**

A program is a *database of facts* (axioms) together with a set of *inference rules* for *proving theorems* from the axioms.

**Imperative Programming:**

☞ Program = Algorithms + Data

**Logic Programming:**

☞ Program = Facts + Rules

or

☞ Algorithms = Logic + Control

# Prolog Facts and Rules

A Prolog program consists of *facts*, *rules*, and *questions*:

*Facts* are named *relations* between objects:

```
parent(charles, elizabeth).
% elizabeth is a parent of charles
female(elizabeth).
% elizabeth is female
```

*Rules* are relations (goals) that can be *inferred* from other relations (subgoals):

```
mother(X, M) :- parent(X,M), female(M).
% M is a mother of X
% if M is a parent of X and M is female
```

# Prolog Questions

*Questions* are statements that can be answered using facts and rules:

```
?- parent(charles, elizabeth).
⇨ yes

?- mother(charles, M).
⇨ M = elizabeth
yes
```

# Horn Clauses

Both *rules* and *facts* are instances of *Horn clauses*, of the form:

$A_0$ if $A_1$ and $A_2$ and ... $A_n$

$A_0$ is the *head* of the Horn clause and "$A_1$ and $A_2$ and ... $A_n$" is the *body*

*Facts* are just Horn clauses without a body:

| | | |
|---|---|---|
| parent(charles, elizabeth) | if | True |
| female(elizabeth) | if | True |
| | | |
| mother(X, M) | if | parent(X,M) |
| | and | female(M) |

# Resolution and Unification

Questions (or *goals*) are answered by *matching* goals against facts or rules, *unifying* variables with terms, and *backtracking* when subgoals fail.

If a subgoal of a Horn clause *matches the head* of another Horn clause, *resolution* allows us to *replace that subgoal* by the body of the matching Horn clause.

*Unification* lets us *bind variables* to corresponding values in the matching Horn clause:

| | | |
|---|---|---|
| | | mother(charles, M) |
| ➪ | | parent(charles, M) and female(M) |
| ➪ | { M = elizabeth } | True and female(elizabeth) |
| ➪ | { M = elizabeth } | True and True |

# Prolog Databases

A *Prolog database* is *a file of facts and rules* to be "consulted" before asking questions:

```
female(anne).
female(diana).
female(elizabeth).

male(andrew).
male(charles).
male(edward).
male(harry).
male(philip).
male(william).
```

```
parent(andrew, elizabeth).
parent(andrew, philip).
parent(anne, elizabeth).
parent(anne, philip).
parent(charles, elizabeth).
parent(charles, philip).
parent(edward, elizabeth).
parent(edward, philip).
parent(harry, charles).
parent(harry, diana).
parent(william, charles).
parent(william, diana).
```

# Simple queries

```
?- consult('royal').
➪ yes
```

*Just another query which succeeds*

```
?- male(charles).
➪ yes

?- male(anne).
➪ no

?- male(mickey).
➪ no
```

…

# Queries with variables

*You may accept or reject unified variables:*

```
?- parent(charles, P).
➪ P = elizabeth <carriage return>
  yes
```

*You may reject a binding to search for others:*

```
?- male(X).
➪ X = andrew ;
  X = charles <carriage return>
  yes
```

*Use anonymous variables if you don't care:*

```
?- parent(william, _).
➪ yes
```

# Unification

Unification is the process of instantiating variables by *pattern matching.*

1. A *constant* unifies only with itself:

```
?- charles = charles.
⇨ yes
?- charles = andrew.
⇨ no
```

2. An *uninstantiated variable* unifies with anything:

```
?- parent(charles, elizabeth) = Y.
⇨ Y = parent(charles,elizabeth) ?
  yes
```

...

# Unification ...

3. A *structured term unifies* with another term only if it has the same function name and number of arguments, and the arguments can be unified recursively:

```
?- parent(charles, P) = parent(X, elizabeth).
➪ P = elizabeth,
  X = charles ?
  yes
```

# Evaluation Order

In principle, any of the parameters in a query may be instantiated or not

```
?- mother(X, elizabeth).
➪ X = andrew ? ;
  X = anne ? ;
  X = charles ? ;
  X = edward ? ;
  no

?- mother(X, M).
➪ M = elizabeth,
  X = andrew ?
  yes
```

# Closed World Assumption

Prolog adopts a *closed world assumption* — whatever cannot be proved to be true, is assumed to be false.

```
?- mother(elizabeth,M).
➪ no

?- male(mickey).
➪ no
```

# Backtracking

Prolog applies resolution in linear fashion, *replacing goals left to right*, and *considering database clauses top-to-bottom*.

```
father(X, M) :- parent(X,M), male(M).
?- trace(father(charles,F)).
⇨ + 1 1 Call: father(charles,_67) ?
  + 2 2 Call: parent(charles,_67) ?
  + 2 2 Exit: parent(charles,elizabeth) ?
  + 3 2 Call: male(elizabeth) ?
  + 3 2 Fail: male(elizabeth) ?
  + 2 2 Redo: parent(charles,elizabeth) ?
  + 2 2 Exit: parent(charles,philip) ?
  + 3 2 Call: male(philip) ?
  + 3 2 Exit: male(philip) ?
  + 1 1 Exit: father(charles,philip) ? ...
```

# Comparison

The predicate = attempts to *unify* its two arguments:

```
?- X = charles.
➪ X = charles ?
  yes
```

The predicate == tests if the terms instantiating its arguments are *literally identical:*

```
?- charles == charles.
➪ yes
?- X == charles.
➪ no
?- X = charles, male(charles) == male(X).
➪ X = charles ?
  yes
```

# Comparison ...

The predicate \== tests if its arguments are *not* literally identical:

```
?- X = male(charles), Y = charles, X \== male(Y).
```

➪ no

# Sharing Subgoals

*Common subgoals* can easily be *factored* out as relations:

```
sibling(X, Y) :- mother(X, M), mother(Y, M),
                 father(X, F), father(Y, F),
                 X \== Y.

brother(X, B) :- sibling(X,B), male(B).
uncle(X, U) :-   parent(X, P), brother(P, U).

sister(X, S) :-  sibling(X,S), female(S).
aunt(X, A) :-    parent(X, P), sister(P, A).
```

# Disjunctions

One may define *multiple rules* for the same predicate, just as with facts:

```
isparent(C, P) :-     mother(C, P).
isparent(C, P) :-     father(C, P).
```

Disjunctions can also be expressed using the ";" operator:

```
isparent(C, P) :-     mother(C, P); father(C, P).
```

Note that *same information* can be represented in *different* forms — we could have decided to express mother/2 and father/2 as facts, and parent/2 as a rule. Ask:

- ❑ Which way is it easier to *express* and *maintain* facts?
- ❑ Which way makes it *faster* to *evaluate* queries?

# Recursion

Recursive relations are defined in the obvious way:

```
ancestor(X, A) :- parent(X, A).
ancestor(X, A) :- parent(X, P), ancestor(P, A).

?- trace(ancestor(X, philip)).
➪ + 1 1 Call: ancestor(_61,philip) ?
  + 2 2 Call: parent(_61,philip) ?
  + 2 2 Exit: parent(andrew,philip) ?
  + 1 1 Exit: ancestor(andrew,philip) ?
X = andrew ?
yes
```

✎ *Will ancestor/2 always terminate?*

# Recursion ...

```
?- trace(ancestor(harry, philip)).
➪ + 1 1 Call: ancestor(harry,philip) ?
  + 2 2 Call: parent(harry,philip) ?
  + 2 2 Fail: parent(harry,philip) ?
  + 2 2 Call: parent(harry,_316) ?
  + 2 2 Exit: parent(harry,charles) ?
  + 3 2 Call: ancestor(charles,philip) ?
  + 4 3 Call: parent(charles,philip) ?
  + 4 3 Exit: parent(charles,philip) ?
  + 3 2 Exit: ancestor(charles,philip) ?
  + 1 1 Exit: ancestor(harry,philip) ?
yes
```

✎ *What happens if you query ancestor(harry, harry)?*

# Evaluation Order

Evaluation of recursive queries is *sensitive to the order of the rules* in the database, and when the recursive call is made:

```
anc2(X, A) :- anc2(P, A), parent(X, P).
anc2(X, A) :- parent(X, A).

?- trace(anc2(harry, X)).
⇨ + 1 1 Call: anc2(harry,_67) ?
  + 2 2 Call: anc2(_325,_67) ?
  + 3 3 Call: anc2(_525,_67) ?
  + 4 4 Call: anc2(_725,_67) ?
  + 5 5 Call: anc2(_925,_67) ?
  + 6 6 Call: anc2(_1125,_67) ?
  + 7 7 Call: anc2(_1325,_67) ? abort
{Execution aborted}
```

# Failure

Searching can be controlled by *explicit failure:*

```
printall(X) :- X, print(X), nl, fail.
printall(_).

?- printall(brother(_,_)).
➪ brother(andrew,charles)
  brother(andrew,edward)
  brother(anne,andrew)
  brother(anne,charles)
  brother(anne,edward)
  brother(charles,andrew)
...
```

# Negation as failure

The *cut* operator (!) *commits* Prolog to a particular search path:

```
parent(C,P) :- mother(C,P), !.
parent(C,P) :- father(C,P).
```

Negation can be implemented by a *combination of cut and fail:*

```
not(X) :- X, !, fail.    % if X succeeds, we fail
not(_).                  % if X fails, we succeed
```

# Changing the Database

The Prolog database can be *modified dynamically* by means of *assert* and *retract:*

```
rename(X,Y) :- retract(male(X)),
               assert(male(Y)), rename(X,Y).
rename(X,Y) :- retract(female(X)),
               assert(female(Y)), rename(X,Y).
rename(X,Y) :- retract(parent(X,P)),
               assert(parent(Y,P)), rename(X,Y).
rename(X,Y) :- retract(parent(C,X)),
               assert(parent(C,Y)), rename(X,Y).
rename(_,_).
```

# Changing the Database ...

```
?- male(charles); parent(charles, _).
➪ yes
?- rename(charles, mickey).
➪ yes
?- male(charles); parent(charles, _).
➪ no
```

*NB: With SICSTUS Prolog, such predicates must be declared dynamic:*

```
:- dynamic male/1, female/1, parent/2.
```

# Functions and Arithmetic

Functions are *relations* between *expressions* and *values*:

```
?- X is 5 + 6.
⇨ X = 11 ?
```

Is *syntactic sugar* for:

```
is(X, +(5,6))
```

# Defining Functions

User-defined functions are written in a *relational style:*

```
fact(0,1).
fact(N,F) :-   N > 0,
               N1 is N - 1,
               fact(N1,F1),
               F is N * F1.
```

```
?- fact(10,F).
⇨ F = 3628800 ?
```

# Lists

**Lists are pairs of elements and lists:**

| *Formal object* | *Cons pair syntax* | *Element syntax* |
|---|---|---|
| .(a , [ ]) | [ a \| [ ] ] | [ a ] |
| .(a , .(b, [ ])) | [ a \| [ b \| [ ] ] ] | [ a , b ] |
| .(a , .(b, .(c , [ ]))) | [ a \| [ b \| [ c \| [ ] ] ] ] | [ a , b, c ] |
| .(a , b) | [ a \| b ] | [ a \| b ] |
| .(a , .(b , c)) | [ a \| [ b \| c ] ] | [ a , b \| c ] |

Lists can be *deconstructed* using cons pair syntax:

```
?- [a,b,c] = [a|X].
➪ X = [b,c]?
```

# Pattern Matching with Lists

```
in(X, [X | _ ]).
in(X, [ _ | L]) :-in(X, L).
```

```
?- in(b, [a,b,c]).
➪ yes
```

```
?- in(X, [a,b,c]).
➪ X = a ? ;
  X = b ? ;
  X = c ? ;
  no
```

# Pattern Matching with Lists ...

Prolog will automatically *introduce new variables* to represent unknown terms:

```
?- in(a, L).
➪ L = [ a | _A ] ? ;
  L = [ _A , a | _B ] ? ;
  L = [ _A , _B , a | _C ] ? ;
  L = [ _A , _B , _C , a | _D ] ?
  yes
```

# Inverse relations

A carefully designed relation can be used in many directions:

```
append([ ],L,L).
append([X|L1],L2,[X|L3]) :- append(L1,L2,L3).

?- append([a],[b],X).
⇨ X = [a,b]

?- append(X,Y,[a,b]).
⇨ X = [] Y = [a,b] ;
  X = [a] Y = [b] ;
  X = [a,b] Y = []
yes
```

# Exhaustive Searching

Searching for permutations:

```
perm([ ],[ ]).
perm([C|S1],S2) :-   perm(S1,P1),
                     append(X,Y,P1), % split P1
                     append(X,[C|Y],S2).
```

```
?- printall(perm([a,b,c,d],_)).
⇨ perm([a,b,c,d],[a,b,c,d])
  perm([a,b,c,d],[b,a,c,d])
  perm([a,b,c,d],[b,c,a,d])
  perm([a,b,c,d],[b,c,d,a])
  perm([a,b,c,d],[a,c,b,d])
...
```

# Limits of declarative programming

A *declarative*, but hopelessly *inefficient* sort program:

```
ndsort(L,S) :-         perm(L,S),
                       issorted(S).
issorted([ ]).
issorted([ _ ]).
issorted([N,M|S]) :-   N =< M,
                       issorted([M|S]).
```

*Of course, efficient solutions in Prolog do exist!*

# What you should know!

- *What are Horn clauses?*
- *What are resolution and unification?*
- *How does Prolog attempt to answer a query using facts and rules?*
- *When does Prolog assume that the answer to a query is false?*
- *When does Prolog backtrack? How does backtracking work?*
- *How are conjunction and disjunction represented?*
- *What is meant by "negation as failure"?*
- *How can you dynamically change the database?*

# Can you answer these questions?

- *How can we view functions as relations?*
- *Is it possible to implement negation without either cut or fail?*
- *What happens if you use a predicate with the wrong number of arguments?*
- *What does Prolog reply when you ask* `not(male(X)).` *? What does this mean?*

# 10. Applications of Logic Programming

**Overview**

- I. Solving a *puzzle*:
  - SEND + MORE = MONEY
- II. Reasoning about *functional dependencies:*
  - finding closures, candidate keys and BCNF decompositions

**References:**


- A. Silberschatz, H.F. Korth and S. Sudarshan, *Database System Concepts*, 3d edition, McGraw Hill, 1997.

# I. Solving a puzzle

✎ *Find values for the letters so the following equation holds:*

```
 SEND
+MORE
-----
MONEY
```

# A non-solution:

We would *like* to write:

```
soln0 :-    A is 1000*S + 100*E + 10*N + D,
            B is 1000*M + 100*O + 10*R + E,
            C is 10000*M + 1000*O + 100*N + 10*E + Y,
            C is A+B,
            showAnswer(A,B,C).

showAnswer(A,B,C) :- writeln([A, ‘ + ‘, B, ‘ = ‘, C]).
writeln([])       :- nl.
writeln([X|L])    :- write(X), writeln(L).
```

# A non-solution ...

```
?- soln0.
➪» evaluation_error: [goal(_1007 is 1000 * _1008 +
  100 * _1009 + 10 * _1010 + _1011),
  argument_index(2)]
  [Execution aborted]
```

But this doesn't work because "is" can only evaluate expressions over *instantiated variables.*

```
?- 5 is 1 + X.
➪» evaluation_error: [goal(5 is
  1+_64),argument_index(2)]
  [Execution aborted]
```

# A first solution

*So let's instantiate them first:*

```
digit(0). digit(1). digit(2). digit(3). digit(4).
digit(5). digit(6). digit(7). digit(8). digit(9).
digits([]).
digits([D|L]):- digit(D), digits(L).

% pick arbitrary digits:
soln1 :- digits([S,E,N,D,M,O,R,E,M,O,N,E,Y]),
         A is 1000*S + 100*E + 10*N + D,
         B is 1000*M + 100*O + 10*R + E,
         C is 10000*M + 1000*O + 100*N + 10*E + Y,
         C is A+B,    % check if solution is found
         showAnswer(A,B,C).
```

# A first solution ...

*This is now correct, but yields a trivial solution!*

```
soln1.
⇨ 0 + 0 = 0
  yes
```

# A second (non-)solution

*So let's constrain S and M:*

```
soln2 :-  digits([S,M]),
          not(S==0), not(M==0), % backtrack if 0
          digits([N,D,M,O,R,E,M,O,N,E,Y]),
          A is 1000*S + 100*E + 10*N + D,
          B is 1000*M + 100*O + 10*R + E,
          C is 10000*M + 1000*O + 100*N + 10*E + Y,
          C is A+B,
          showAnswer(A,B,C).
```

# A second (non-)solution ...

Maybe it works. We'll never know ...

```
soln2.
➪   [Execution aborted]
```

*after 8 minutes still running ...*

✎ *What went wrong?*

# A third solution

Let's try to exercise more control by *instantiating variables bottom-up:*

```
sum([],0).
sum([N|L], TOTAL) :- sum(L,SUBTOTAL),
                     TOTAL is N + SUBTOTAL.

% Find D and C, where ∑L is D + 10*C, digit(D)
carrysum(L,D,C) :-
        sum(L,S), C is S/10, D is S - 10*C.

?- carrysum([5,6,7],D,C).
➪ D = 8
  C = 1
```

# A third solution ...

We instantiate the final digits first, and use the carrysum to *constrain the search space:*

```
soln3 :- digits([D,E]), carrysum([D,E],Y,C1),
         digits([N,R]), carrysum([C1,N,R],E,C2),
         digit(O), carrysum([C2,E,O],N,C3),
         digits([S,M]), not(S==0), not(M==0),
         carrysum([C3,S,M],O,M),
         A is 1000*S + 100*E + 10*N + D,
         B is 1000*M + 100*O + 10*R + E,
         C is A+B,
         showAnswer(A,B,C).
```

# A third solution ...

*This is also correct, but uninteresting:*

```
soln3.
⇨  9000 + 1000 = 10000
   yes
```

# A fourth solution

Let's try to make the variables *unique:*

```
% There are no duplicate elements in the argument list
unique([X|L]) :- not(in(X,L)), unique(L).
unique([]).

in(X, [X|_]).
in(X, [_|L]) :- in(X, L).

?- unique([a,b,c]).
⇨ yes
?- unique([a,b,a]).
⇨ no
```

# A fourth solution ...

```
soln4 :- L1 = [D,E], digits(L1), unique(L1),
         carrysum([D,E],Y,C1),
         L2 = [N,R,Y|L1], digits([N,R]), unique(L2),
         carrysum([C1,N,R],E,C2),
         L3 = [O|L2], digit(O), unique(L3),
         carrysum([C2,E,O],N,C3),
         L4 = [S,M|L3], digits([S,M]),
           not(S==0), not(M==0), unique(L4),
         carrysum([C3,S,M],O,M),
         A is 1000*S + 100*E + 10*N + D,
         B is 1000*M + 100*O + 10*R + E,
         C is A+B,
         showAnswer(A,B,C).
```

# A fourth solution ...

*This works (at last), in about 1 second on a G3 Powerbook.*

```
soln4.
⇨   9567 + 1085 = 10652
    yes
```

# II. Reasoning about functional dependencies

We would like to represent *functional dependencies* for relational databases as Prolog terms, and write predicates that compute:

(i) *closures* of attribute sets,

(ii) *candidate keys*, and

(iii) *BCNF* decompositions.

# Operator overloading

First, we would like to overload Prolog syntax as follows:

```
FDS = [ [a]->[b,c], [c,g]->[h,i], [b,c]->[h] ].
```

➪ *Syntax Error - unable to parse » ->[b,c] ...*

but the built-in arrow operator has precedence higher than that of "," and "=":

```
op(1050, xfy, [ -> ]).
op(1000, xfy, [ ',' ]).
op(700, xfx, [ = ]).
```

so let's change it:

```
:- op(600, xfx, [ -> ]).
```

*Now we can get started ...*

# Computing closures

We would like to define a predicate:

```
closure(FDS, AS, CS)
```

which computes the closure *CS* of an attribute set *AS* using the dependencies in FDS.

```
?- closure([[a]->[b], [b]->[c]], [a], Closure).
➪ Closure = [b,a,c]
```

# Computing closures ...

We should use Armstrong's axioms:

| | | | | |
|---|---|---|---|---|
| 1. | $B \subseteq A$ | $\Rightarrow$ | $A \rightarrow B$ | (reflexivity) |
| 2. | $A \rightarrow B$ | $\Rightarrow$ | $AC \rightarrow BC$ | (augmentation) |
| 3. | $A \rightarrow B$, $B \rightarrow C$ | $\Rightarrow$ | $A \rightarrow C$ | (transitivity) |

Intuitively, we add attributes to a set AS′, using the axioms and the FDs, until no more dependencies can be applied:

- start with $AS \rightarrow AS'$, where $AS' = AS$ (1)
- find some $B \rightarrow C$, $AS' = BD \Rightarrow AS \rightarrow AS' \rightarrow CD$ (2,3)
- repeat till no more FD applies

*NB: each FD can be applied at most once!*

# A closure predicate

We try to express the algorithm *declaratively:*

```
closure(FDS, AS, CS) :-
  applies(FDS, B->C, AS, FDRest), !,  % NB cut
  union(AS, C, AS1),
  closure(FDRest, AS1, CS).
closure(FDS, AS, AS).         % no more FD applies

applies(FDS, B->C, AS, FDRest) :-
  in(B->C, FDS), rem(B->C, FDS, FDRest),
  subset(B,AS).
```

*Now we must worry about the details ...*

# Manipulating sets

We need some predicates to manipulate attribute sets and sets of FDs:

```
in(X, [X|_]). % in(X,S) -- X is in the argument list
in(X, [_|S]) :- in(X, S).

subset([],_). % subset(S1,S2) -- S1 is a subset of S2
subset([X|S1],S2) :- in(X,S2), subset(S1,S2).

rem(_,[],[]). % rem(X,S,R) -- S\{X} yields R
rem(X,[X|S],R) :- rem(X,S,R), !.
rem(X,[Y|S],[Y|R]) :- rem(X,S,R) .
```

...

✎ *How would you express set union and intersection?*

# Evaluating closures

```
?- FDS = [ [a]->[b,c],
           [c,g]->[h,i],
           [b,c]->[h]
         ],
closure(FDS, [a], Ca),
closure(FDS, [a,c], Cac),
closure(FDS, [a,g], Cag).
```

```
➪ FDS = [[a]->[b,c],[c,g]->[h,i],[b,c]->[h]]
  Ca = [c,b,a,h]
  Cac = [b,a,c,h]
  Cag = [i,h,g,a,b,c]
  yes
```

# Testing

We cast all our examples as test cases:

```
testClosures :-
  FDS = [[a]->[b,c], [c,g]->[h,i], [b,c]->[h] ],
  closure(FDS, [a], Ca),
  check('closure[a]', equal(Ca, [a,b,c,h])),
  ...

check(Name, Goal) :-
  Goal, !.
check(Name, Goal) :-
  writeln([Name, ' FAILED']).
```

# Finding keys

Now we would like a predicate `candkey/2` that suggests a candidate key for the attributes in a set of FDs:

```
candkey(FDS, Key) :-
    attset(FDS, AS), % get the complete attribute set
    minkey(FDS, AS, AS, Key).
```

*Given Key -> AS, search for the smallest MinKey -> AS*

```
minkey(FDS, AS, Key, MinKey) :-
    smallerkey(FDS, AS, Key, SmallerKey), !,
    minkey(FDS, AS, SmallerKey, MinKey).
minkey(FDS, AS, MinKey, MinKey).
```

✎ *How would you implement* `attset/2`*?*

# Finding keys ...

*A smaller key is smaller, and is still a key!*

```
smallerkey(FDS, AS, Key, Smaller) :-
  in(X, Key),
  rem(X, Key, Smaller),
  iskey(Smaller, AS, FDS).
```

*Key -> AS if AS $\subseteq$ $K^+$*

```
iskey(Key, AS, FDS) :-
  closure(FDS, Key, Closure),
  subset(AS, Closure).
```

# Evaluating candidate keys

```
?- FDS = [[a]->[b,c],[c,g]->[h,i],[b,c]->[h]],
candkey(FDS, Key).
```

➪ Key = [a,g]

```
?- FDS = [[name]->[addr],[name,article]->[price]],
candkey(FDS, Key).
```

➪ Key = [name,article]

# Testing for BCNF

A relation scheme is in BCNF if *all non-trivial FDs define keys:*

```
isbcnf(FDS, RS) :- fdsok(FDS, FDS, RS).

fdsok([A->B|ToCheck], FDS, RS) :-
  subset(B,A),                       % A->B is trivial
  fdsok(ToCheck,FDS,RS).
fdsok([A->B|ToCheck], FDS, RS) :-
  subset(A, RS), !,                  % A applies to RS
  iskey(A, RS, FDS),                 % A is a key for RS
  fdsok(ToCheck,FDS,RS).
fdsok([A->B|ToCheck], FDS, RS) :-
  fdsok(ToCheck,FDS,RS).             % A doesn't apply
fdsok([], _, RS).                    % Done checking
```

# Evaluating the BCNF test

```
?- FDS = [[name]->[addr], [name, article]->[price]],
   isbcnf(FDS, [name, addr]),
   not(isbcnf(FDS, [name, article, price])),
   not(isbcnf(FDS, [name, addr, article, price])).
```

➪ yes

```
?- FDS = [[city, street] -> [zip], [zip] -> [city]],
   attset(FDS, As),
   isbcnf(FDS, As).
```

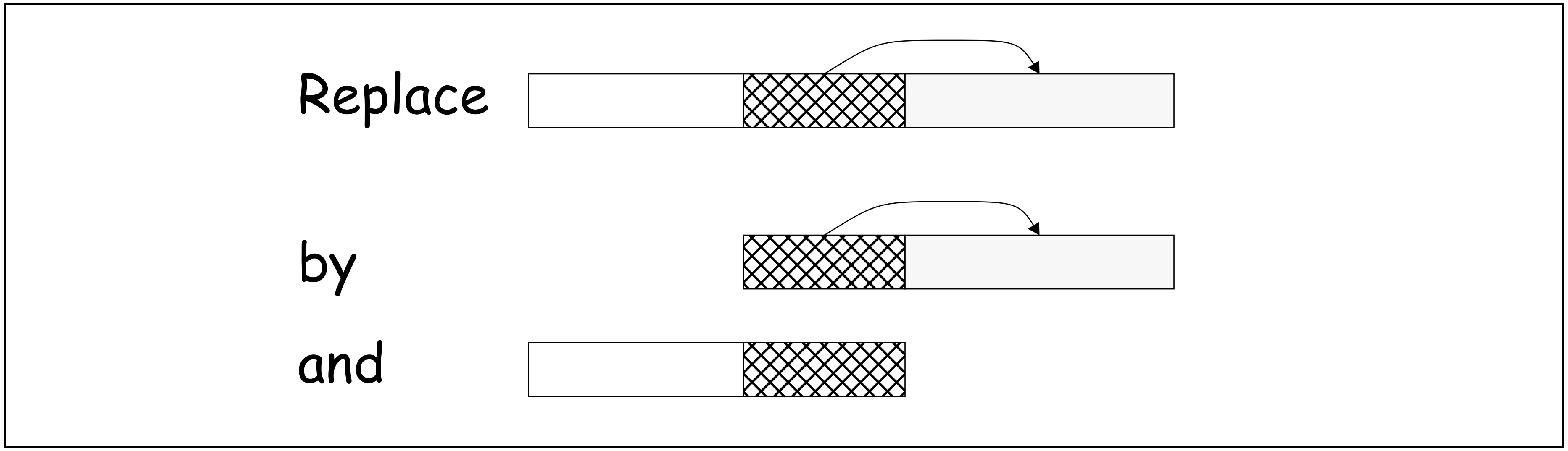
➪ no

✎ *How can we find out exactly which FD is problematic?*

# BCNF decomposition

Recall that BCNF decomposition works as follows:

while some R is not in BCNF
    select non-trivial $\alpha \rightarrow \beta$ holding on R where
        $\alpha \rightarrow R$ is not in $F^+$ and $\alpha \cap \beta = \varnothing$
    replace R by $\alpha \cup \beta$ and (R-$\beta$)




The trick is that $\alpha \rightarrow \beta$ may not be explicitly in the list F of FDs, and it is too expensive to compute the closure $F^+$

# BCNF decomposition — top level

We start decomposing with the full attribute set:

```
bcnf(FDS, Decomp) :-
  attset(FDS, AS),
  bcnfDecomp(FDS, [AS], Decomp).
```

# BCNF decomposition — recursion

We must iterate through *both* the FDS *and* the schema.

*RS not in BCNF, so decompose:*

```
bcnfDecomp(FDS, [RS|Schema], Decomp) :-
    findBad(A->B, FDS, FDS, RS),
    union(A,B,AB),
    diff(RS,B,Diff),
    bcnfDecomp(FDS, [AB,Diff|Schema], Decomp).
```

*RS is OK, so accept it and recurse:*

```
bcnfDecomp(FDS, [RS|Schema], [RS|Decomp]) :-
    bcnfDecomp(FDS, Schema, Decomp).
```

*Nothing left to do:*

```
bcnfDecomp(FDS, [], []).
```

# Finding "bad" FDs

The "bad" FDs may be in the *closure* the given FDs.

```
findBad(A->B, [FD|FDS], AllFDS, RS) :- % A->B is bad
  FD = A->B0,                          % Try to derive a bad FD
  subset(A,RS),                        % A must apply to RS
  diff(B0,A,B1),                       % A ∩B should be empty
  inter(B1,RS,B),                      % restrict to RS
  not(subset(B,A)),                    % FD must not be trivial
  not(iskey(A, RS, AllFDS)).% "bad" if A is not a key

findBad(FD, [OK|FDS], AllFDS, RS) :-
  findBad(FD, FDS, AllFDS, RS).
```

✎ *Can you justify this derivation using Armstrong's axioms?*

# Evaluating BCNF decomposition

```
?- FDS = [[name]->[addr],[name,article]->[price]],
   bcnf(FDS, BCNF).
```

➪ `BCNF = [[name,addr],[name,price,article]]`

```
?- FDS = [[city,street]->[zip],[zip]->[city]],
   bcnf(FDS, BCNF).
```

➪ `BCNF = [[zip,city],[zip,street]]`

✎ *What would you have to change in order to find all BCNF decompositions?*

# Can you answer these questions?

- *What happens when we ask* `digits([A,B,A])`*?*
- *How many times will* `soln2` *backtrack before finding a solution?*
- *How would you check if the solution to the puzzle is unique?*
- *How would you generalize the puzzle solution to solve arbitrary additions?*
- *Can you use subset/2 to find all subsets of a set?*
- *Will all the recursive predicates terminate?*
- *What would happen if we didn't cut in* `minkey/4`*?*
- *How could we generate the set of all min keys?*
- *Would it be just as easy to implement these solutions with a functional language?*

# 11. Symbolic Interpretation

**Overview**

- Interpretation as Proof
- Operator precedence: representing programs as syntax trees
- An interpreter for the calculator language
- Implementing a Lambda Calculus interpreter
- Examples of lambda programs ...

# Interpretation as Proof

One can view the execution of a program as a step-by-step *"proof"* that the program *reaches some terminating state*, while producing output along the way.

- The *program* and its intermediate states are represented as *structures* (typically, as syntax trees)
- *Inference rules* express how one program state can be *transformed* to the next

# Representing Programs as Trees

Recall our Calculator example [Schmidt]:

```
P   ::=    'on' S
S   ::=    E 'total' S      |   E 'total' 'OFF'
E   ::=    E1 '+' E2        |   E1 '*' E2
       |   'if' E1 'then' E2 'else' E3
       |   'lastanswer'     |   '(' E ')'| N
```

*Syntax trees* can be modelled directly as *Prolog terms.*

For example, the program:

```
on 2+3 total lastanswer + 1 total off
```

can be modelled by the term:

```
on(total(2+3, total(lastanswer+1, off)))
```

# Prefix and Infix Operators

Operator type and precedence can be defined to achieve convenient syntax:

```
:- op(900,fx,on).       % prefix
:- op(800,xfy,total).   % right assoc.
:- op(600,fx,if).
:- op(590,xfy,then).
:- op(580,xfy,else).
% op(500,yfx,+).        % left assoc.
% op(400,yfx,*).        % pre-defined ...
```

The higher the precedence, the higher in the syntax tree the operator will appear.

# Prefix and Infix Operators ...

Operators can be declared:

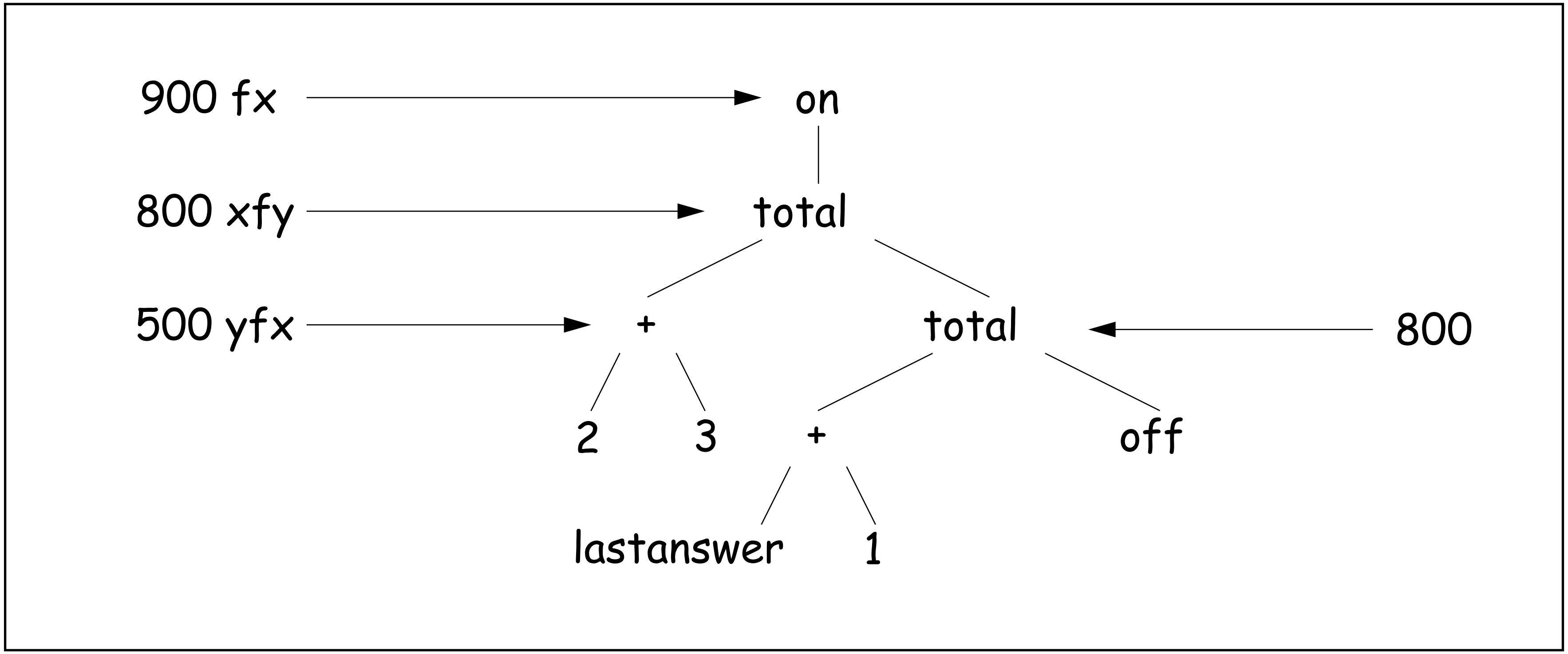
- (i) *xfy* for *right-associative*, (e.g., ;)
- (ii) *yfx* for *left-associative*, (e.g., +)
- (iii) *xfx* for *non-associating*, (e.g. =)
- (vi) *fx* and *fy* for *prefix*, (e.g., `not not P`)
- (v) *xf* and *yf* for *postfix*

```
?- 1+2+3*4 = +(+(1,2),*(3,4)).
➪ yes
```

```
?- (on 2+3 total lastanswer+1 total off)
   == on(total(2+3, total(lastanswer+1, off))).
➪ yes
```

# Operator precedence

```
on 2+3 total lastanswer+1 total off
  == on(total(2+3, total(lastanswer+1, off))).
```



# Standard Operators

The following operator precedences are predefined for SICSTUS Prolog:

```
op(1200,xfx, [ :- , -- ]).
op(1200,fx,  [ :- , ?- ]).
op(1150,fx,  [ mode , public , dynamic , multifile , parallel , wait ]).
op(1100,xfy, [ ; ]).
op(1050,xfy, [ -> ]).
op(1000,xfy, [ ',' ]).
op(900, fy,  [ \+ , spy , nospy ]).
op(700, xfx, [ =, is, =.., ==, \==, @<, @>, @=<, @>=, =:=, =\=, <, >,
                =<, >= ]).
op(500, yfx, [ +, - , /\ , \/ ]).
op(500, fx,  [ + , - ]).
op(400, yfx, [ * , / , // , << , >> ]).
op(300, xfx, [ mod ]).
op(200, xfy, [ ^ ]).
```

# Building a Simple Interpreter

*We define semantic predicates over the syntactic elements of our calculator language.*

**Top level:**

```
on S            :- peval(S, L), write(L).
```

**Programs:**

```
peval(S,L)      :- seval(S, 0, L).
```

**Statements:**

```
seval(E total off, Prev, [Val]) :-
    xeval(E, Prev, Val).

seval(E total S, Prev, [Val|L]) :-
    xeval(E, Prev, Val),
    seval(S, Val, L).
```

# Building a Simple Interpreter ...

**Expressions:**

```
xeval(N, _, N) :- number(N).
xeval(lastanswer, Prev, Prev).

xeval(if E1 then E2 else _, Prev, Val) :-
  xeval(E1, Prev, 0),
  xeval(E2, Prev, Val).

xeval(if E1 then _ else E3, Prev, Val) :-
  xeval(E1, Prev, V1), V1 =\= 0,
  xeval(E3, Prev, Val).
...
```

✎ *Can you fill in the missing cases?*

# Running the Interpreter

```
?- on 2+3 total lastanswer+1 total off.
➪ [5,6] yes
```

# Lambda Calculus Interpreter

*Now a more ambitious example ..*

First we must choose a syntax for lambda expressions:

```
:- op(650, xfy, :).     % body of abstraction
:- op(600, fx, \).      % abstraction
:- op(500, yfx, @).     % application
```

Unfortunately, we cannot write `e1 e2` in Prolog, so we must introduce an *operator* for *application*.

For example, we will represent the lambda expression:

$$(\lambda x\ .\ \lambda y\ .\ x\ y)\ y$$

by the Prolog term:

```
(\x: \y: x@y) @ y == @(:(\(x),:(\(y),@(x,y))), y).
```

# Semantics

Alpha, beta and eta conversion are expressed as predicates over the "*before*" and "*after*" forms of lambda expressions:

```
alpha(\X:E, \Y:EY) :-
                fv(E, FE),
                not(in(Y, FE)),
                subst(Y, X, E, EY).
beta((\X:E1)@E2, E3) :-
                subst(E2, X, E1, E3).
eta(\X:E@X, E) :-
                fv(E, F),
                not(in(X, F)).
```

# Free Variables

To implement *conversion* and *reduction*, we need to know the free variables in an expression:

```
fv(X, [X]) :-       isname(X).

fv(E1@E2, F12) :-  fv(E1, F1),
                    fv(E2, F2),
                    union(F1, F2, F12).

fv(\X:E, F) :-      isname(X),
                    fv(E, FE),
                    diff(FE, [X], F).

isname(N) :-        atom(N); number(N).
```

# Free Variables ...

*For example:*

```
?- fv(\x: \y:x@y@z , F).
➪ F = [z] ?
  yes
```

# Substitution

subst(E, X, EX, EE) substitutes E for X in EX, yielding EE:

```
subst(E, X, X, E) :-      isname(X), !.
subst(E, X, Y, Y) :-      isname(X), isname(Y),
                          X \== Y.
subst(E, X, E1@E2, EE1@EE2) :-
                          subst(E, X, E1, EE1),
                          subst(E, X, E2, EE2).
subst(E, X, \X:E1, \X:E1).
subst(E, X, \Y:E1, \Y:EE1) :-
                          X \== Y,
                          fv(E, FE),
                          not(in(Y, FE)), !,
                          subst(E, X, E1, EE1).
```

# Avoiding name capture

We avoid *name capture* by substituting Y by a *new name* Z:

```
subst(E, X, \Y:E1, \Z:EEZ) :-X \== Y,
                            fv(E, FE),
                            % in(Y, FE),
                            fv(E1, F1),
                            union(FE, F1, FU),
                            newname(Y, Z, FU),
                            subst(Z, Y, E1, EZ),
                            subst(E, X, EZ, EEZ).
```

# Renaming

newname(Y, Z, F) is true if Z is a new name for Y, not in F

```
newname(Y, Y, F) :- not(in(Y, F)), !.
newname(Y, Z, F) :- tick(Y, T), newname(T, Z, F).
```

The built-in predicate name(X, L) is true if the name X is represented by the ASCII list L

tick(Y, Z) is true if Z is Y with a “tick” (' = ASCII 39) appended

```
tick(Y, Z) :- name(Y, LY),
              append(LY, [39], LZ),
              name(Z, LZ).
```

# Renaming ...

*For example:*

```
?- tick(x, Y).
➪ Y = x′ ?
  yes

?- subst(x@y, z, \x:x@z, E).
➪ E = \x':x'@(x@y)
  yes
```

# Normal Form Reduction

E => NF is true if E *reduces to normal form* NF;
lazy(E, EE) is true if E reduces to EE by *one* normal-order reduction:

```
:- op(900, xfx, =>).
E => NF :-      lazy(E, EE), !, EE => NF.
X => X.         % no more reductions possible, so stop

lazy(E1, E2) :-           beta(E1, E2), !.
lazy(E1, E2) :-           eta(E1, E2), !.
lazy(E0@E2, E1@E2) :-     lazy(E0, E1), !.
```

✎ *What happens if you leave out the third lazy/2 rule?*
✎ *How would you change this to be strict evaluation?*

# Normal Form Reduction ...

*For example:*

```
?- (\x : (\y:x)@(\x:x)@x ) @ y => E.
➪ E = y@y ?
  yes
```

# Viewing Intermediate States

The => predicate tells us what normal form a lambda expression reduces to, but does not tell us *which reductions* take us there.

To see intermediate reductions, we can print out each step:

```
:- op(800, fx, eval).
eval E :-       lazy(E, EE), !,
                write(E), nl, write('-> '),
                eval EE.
eval E :-       write(E), nl, write('STOP'), nl.
```

✎ *Can you think of other ways to solve this problem?*

# Viewing Intermediate States ...

*The same example yields:*

```
?- eval (\x: \y: x@y) @ y.
⇨ (\x: \y:x@y)@y
    -> \y':y@y'
    -> y
    STOP
```

# Lazy Evaluation

Recall that the lambda expression $\Omega = (\lambda x . x\ x)(\lambda x . x\ x)$ *has no normal form:*

```
?- W = ((\x:x@x) @ (\x:x@x)),
   eval W.
⇨ (\x:x@x)@(\x:x@x)
     -> (\x:x@x)@(\x:x@x)
     -> (\x:x@x)@(\x:x@x)
     <interrupt>
[Execution aborted]
```

# Lazy Evaluation ...

*But lazy evaluation allows it to be passed as a parameter if unused!*

```
?- W = ((\x:x@x) @ (\x:x@x)),
   eval (\x:y) @ W.
➪ (\x:y)@((\x:x@x)@(\x:x@x))
     -> y
     STOP
```

# Booleans

Recall the standard encoding of Booleans as lambda expressions that return their first (or second) argument:

```
?- True = \x: \y:x,
   False = \x: \y:y,
   Not = \b:b@False@True,
   eval Not@True.
➪ (\b:b@(\x: \y:y)@(\x: \y:x))@(\x: \y:x)
     -> (\x: \y:x)@(\x: \y:y)@(\x: \y:x)
     -> (\y: \x: \y:y)@(\x: \y:x)
     -> \x: \y:y
     STOP
```

# Tuples

Recall that tuples can be modelled as *higher-order functions* that pass the values they hold to another (client) function:

```
?- True = \x: \y:x, False = \x: \y:y,
   Pair = (\x: \y: \z: z@x@y),
   First = (\p:p @ True),
   eval First @ (Pair @ 1 @ 2).
⇨ (\p:p@(\x: \y:x))@((\x: \y: \z:z@x@y)@1@2)
     -> (\x: \y: \z:z@x@y)@1@2@(\x: \y:x)
     -> (\y: \z:z@1@y)@2@(\x: \y:x)
     -> (\z:z@1@2)@(\x: \y:x)
     -> (\x: \y:x)@1@2
     -> (\y:1)@2
     -> 1
     STOP
```

# Natural Numbers

And natural numbers can be modelled using the standard encoding:

```
?- True = \x: \y:x, False = \x: \y:y,
   Pair = (\x: \y: \z: z@x@y),
   First = (\p:p @ True),
   Second = (\p:p @ False),
   Zero = \x:x,
   Succ = \n:Pair@False@n,
   Succ@Zero => One,
   IsZero = First,
   Pred = Second,
   eval IsZero@(Pred@One).
```

# Natural Numbers ...

*Though you probably won't like what you see!*

```
➪ (\p:p@(\x: \y:x))@((\p:p@(\x: \y:y))
        @(\z:z@(\x: \y:y)@(\x:x)))
    -> (\p:p@(\x: \y:y))
        @(\z:z@(\x: \y:y)@(\x:x))@(\x: \y:x)
    -> (\z:z@(\x: \y:y)@(\x:x))@(\x: \y:y)@(\x: \y:x)
    -> (\x: \y:y)@(\x: \y:y)@(\x:x)@(\x: \y:x)
    -> (\y:y)@(\x:x)@(\x: \y:x)
    -> (\x:x)@(\x: \y:x)
    -> \x: \y:x
    STOP
yes
```

# Fixed Points

Recall that we could not model the fixed point combinator Y in Haskell because *self-application cannot be typed.*

In our untyped interpreter, we can implement Y:

```
?- Y = \f:(\x:f@(x@x))@(\x:f@(x@x)),
   FP = Y@e,
   eval FP.
⇨ (\f:(\x:f@(x@x))@(\x:f@(x@x)))@e
    -> (\x:e@(x@x))@(\x:e@(x@x))
    -> e@((\x:e@(x@x))@(\x:e@(x@x)))
    STOP
```

*Note that this sequence validates that e@FP <-> FP.*

# Recursive Functions as Fixed Points

```
?- True = \x: \y:x, False = \x: \y:y,
   Pair = (\x: \y: \z: z@x@y),
   First = (\p:p @ True), Second = (\p:p @ False),
   Zero = \x:x, Succ = \n:Pair@False@n,
   Succ@Zero => One,
   IsZero = First, Pred = Second,
   Y = \f:(\x:f@(x@x))@(\x:f@(x@x)),
   RPlus = \plus: \n: \m :
     IsZero@n @m @(plus @ (Pred@n)@(Succ@m))
   Y@RPlus => FPlus, FPlus@One@One => Two,
   eval IsZero@(Pred@(Pred@Two)).
```

# Recursive Functions as Fixed Points ...

```
➪ (\p:p@(\x: \y:x))@((\p:p@(\x: \y:y))@((\p:p@(\x: \y:y))
       @(\z:z@(\x: \y:y)@(\z:z@(\x: \y:y)@(\x:x)))))
  -> (\p:p@(\x: \y:y))@((\p:p@(\x: \y:y))@(\z:z@(\x: \y:y)
       @(\z:z@(\x: \y:y)@ (\x:x)))))@ (\x: \y:x)
  -> (\p:p@(\x: \y:y)) @ (\z:z@(\x: \y:y)@(\z:z@(\x: \y:y)@(\x:x)))
       @ (\x: \y:y)@(\x: \y:x)
  -> (\z:z@(\x: \y:y)@(\z:z@(\x: \y:y)@(\x:x)))@(\x: \y:y)
       @(\x: \y:y)@(\x: \y:x)
  -> (\x: \y:y)@(\x: \y:y)@(\z:z@(\x: \y:y)@(\x:x))@(\x: \y:y)
       @(\x: \y:x)
  -> (\y:y)@(\z:z@(\x: \y:y)@(\x:x))@(\x: \y:y)@(\x: \y:x)
  -> (\z:z@(\x: \y:y)@(\x:x))@(\x: \y:y)@(\x: \y:x)
  -> (\x: \y:y)@(\x: \y:y)@(\x:x)@(\x: \y:x)
  -> (\y:y)@(\x:x)@(\x: \y:x)
  -> (\x:x)@(\x: \y:x)
  -> \x: \y:x
  STOP
```

# What you should know!

- *How can you represent programs as syntax trees?*
- *How can you represent syntax trees as Prolog terms?*
- *How can you define the syntax of your own language in Prolog?*
- *Why did we define ":" as right-associative but "@" as left-associative?*
- *What is the difference between* `Succ@Zero=>One` *and* `One=Succ@Zero`*?*

# Can you answer these questions?

- *How would you implement an interpreter for the assignment language we defined earlier?*
- *Why didn't we use "." in our syntax for lambda expressions?*
- *Does the order of the* `fv/2` *rules matter? What about* `subst/4`*?*
- *Can you explain each usage of "cut" (!) in the lambda interpreter?*
- *Can you think of other ways to implement* `newname/3`*?*
- *How would you modify the lambda interpreter to use strict evaluation?*